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セミハイパーリングと添加セミハイパーリング

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あらまし　適当な固定した k で，ノードの集合は $V = \{k, k+1, ..., P-k-1\}$, 辺の集合は $E = \{\{u,v\}|u$ と v の間の位置的な距離は２の幂乗$\}$ であるとき，グラフ $G = (V, E)$ を P 個のノードをもつセミハイパーリング (P-SHR と略記) という．２の冪乗ではない，適当な固定した $3 \leqq r < P$ で，P-SHR に位置的な距離が r であるような全てのノードの組の間に辺を加えたとき，そのグラフを添加セミハイパーリングという．P-SHR(添加 P-SHR) の連結度は P-SHR(添加 P-SHR) の次数に等しいことを，そのグラフの任意のノードの組の間に，その次数に等しい個数の点素なパスを構成することにより示す．

キーワード　連結度, ハイパーリング, ネットワーク, 点素なパス, セミハイパーリング

Semi Hyper-rings and Enriched Semi Hyper-rings

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Abstract　A graph $G = (V,E)$ is called a semi hyper-ring with P nodes (P-SHR for short) if $V = \{k, k+1, ..., P+k-1\}$ for some fixed k and $E = \{\{u,v\}|$ the positional distance between nodes u and v is a power of $2\}$. An enriched P-SHR is a graph constructed from P-SHR by allowing additional connections between all pairs of nodes that are k positional distance away each other, for a fixed k, $3 \leq k < P$, which is a not power of 2. We show that the connectivity of a P-SHR (an enriched P-SHR) is equal to its degree, say δ, by the explicit construction of δ node-disjoint paths connecting nodes s and t.

Key words　Connectivity, hyper-ring, network, node disjoint path, semi hyper-ring

1. Introduction

Various topologies have been proposed for interconnecting processors in large scaled parallel and/or distributed systems. Hyper-rings (HRs) [2] and their variations have appeared in the literature under several names, including optimal broadcasting scheme [1] and binary jumping networks [9]. Detailed discussions on the structure, properties, and advantages of this family of networks and their fault tolerance were presented in [2], [8], [9].

Node-connectivity of a network is fundamental in the analysis of network reliability and/or security. Menger's Theorem relates the connectivity of a graph G, denoted by $\kappa(G)$, to the size of the smallest set (among the maximal sets) of pair-wise internally node-disjoint paths between any pair of nodes s and t [13]. Note that in any G, $\kappa \leq \delta$, where δ denotes the minimal nodal degree of G. Hence, in a network, multiple copies of a message may be sent through a number of disjoint paths and fault tolerance can be achieved in this manner [11]. In addition to increased reliability, disjoint paths provide an excellent mechanism for transmitting secret messages through a network. Using Rabin's Information Dispersal Algorithm [12], or partitioning a secret message into submessages, where every i-th packet is transmitted via the i-th mod κ path, may prevent (or at least make it more difficult) for adversaries to intercept and decipher it.

Semi hyper-rings and enriched semi hyper-rings were introduced by Altman et al. [4]. Each of a semi hyper-ring and an enriched semi hyper-ring is a subgraph of a hyper-ring. The structures of these graphs are closely related to the structure of a hyper-ring. Semi hyper-rings and enriched semi hyper-rings play an important role in the construction of node disjoint paths in a hyper-ring [4]. We show that the connectivity of a semi hype-ring (an enriched semi hyper-ring) is equal to its degree, say δ, by the explicit construction of δ node-disjoint paths connecting nodes s and t.

2. Preliminaries

Let us begin with a formal definition of a hyper-ring.

[Definition 1] *A graph* $G = (V, E)$ *is called a* hyper-ring *with N nodes (N-HR for short) if* $V = \{0, ..., N-1\}$ *and* $E = \{\{u, v\} | [v-u]_N$ *is a power of 2* $\}$, *where* $[m]_r$ *is m modulo r.*

[Definition 2] *The* positional distance *between node s and node t of N-HR is defined to be* $min\{[t-s]_N, [s-t]_N\}$.

The number of edges between nodes in an HR is roughly twice that of a hyper cube (HC for short) with the same number of nodes, but the proposed organization possesses a number of advantages over the HC. In particular, for any N we can construct an N node HR, whereas a hypercube must contain exactly 2^k nodes for some k. An example of an 11-HR is shown in Figure 1.

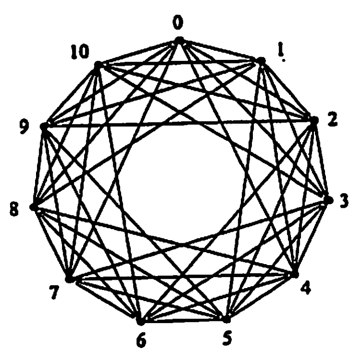

図 1 An example of an 11-HR.

Note that what appears to be a connection of positional distance 3 is really a (counterclockwise) connection of positional distance 8.

The HCs have a natural and an elegant recursive structure that does not seem to extend to HRs in an obvious way. In particular, the number of edges in HRs does not grow monotonically with the number of nodes [2]. It turns out, however, that HRs do possess an interesting recursive construction that encompasses HCs in a sense as a special case. In order to study the recursive structure of HRs, we will need to examine the structure of N itself.

First, let us observe a simple (nonrecursive) construction of N-HRs. The following procedure is straightforward and needs no further explanation.

```
for i := 0 to N − 1 do
    for k := 0 to ⌊log N⌋ do
        if there is no edge between i and [i + 2^k]_N
            then connect node i to nodes labeled [i ± 2^k]_N
```

Observe that in HRs, the two edges usually connecting nodes that are either $2^{\lfloor \log N \rfloor}$ or $2^{\lfloor \log N \rfloor - 1}$ positional distance away from each other either do not exist as such (examine the $(2^i + 2^j)$-HR family), may merge into one (2^k-HR family), or crisscross (all of the remaining HRs). Again, see edges (0,8) and $(0, [-8]_{11} = 3)$ in Figure 1.

Assume that an N-HR has already been constructed. Suppose we wish to construct a $2N$-HR from it. The following procedure [2], takes as input an N-HR_0 and returns a $2N$-HR.

```
make a duplicate copy of N-HR_0, call it N-HR_1
for i := 0 to N − 1 do
    relabel node i in N-HR_0 to 2i
    relabel node i in N-HR_1 to 2i + 1
for i := 0 to 2N − 1 do
    connect node i and node [i + 1]_2N
```

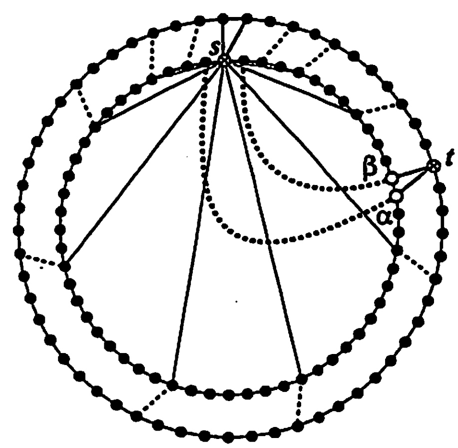

図 2 An HR constructed using the doubling procedure and the two additional node-disjoint paths via α and β.

Figure 2 shows the connections for one node of the HR constructed using the doubling construction procedure. It shows that if δ is the degree of an N-HR, $N > 1$, then the degree of $2N$-HR is $\delta + 2$. Previously, it was shown that the doubling of an HR increases its connectivity from κ to $\kappa + 2$ [3]. The essence of the proof is captured by the two paths containing nodes α and β in Figure 2, however, it will not be discussed here.

We now introduce formal definitions of a semi hyper-ring and an enriched semi hyper-ring.

[Definition 3] *A semi hyper-ring is a graph corresponding to any segment of P consecutive integers (P-SHR for short) in which any two nodes (integers) are connected if and only if they are a power of 2 positional distance away each other.*

For any $t \geq 0$, a P-SHR with P consecutive nodes (integers) starting from node t and ending at node $t + P - 1$ is isomorphic to any P-SHR if they are segments of P consecutive nodes in the same N-HR ($P \leq N$). In particular, for any $t \geq 0$, a P-SHR with P consecutive nodes (integers) starting from node t is isomorphic a P-SHR with P consecutive nodes (integers) starting from node 0.

[Definition 4] *A P-SHR is said to be k-enriched (k-enriched P-SHR) if it is extended by allowing additional connections between all pairs of nodes that are r positional distance away from each other, for a fixed r, $3 \leq r < P$, which is not a power of 2.*

3. Node connectivity of SHRs and enriched SHRs

We first show that the connectivity of P-SHR is equal to its degree.

[Theorem 1] *The node-connectivity of a P-SHR is equal to $\lceil \log P \rceil$, which is also the degree of its first and last nodes.*

Proof. It is obvious that the degree of the first and last nodes of any P-SHR is $\lceil \log P \rceil$. Observe that for all $k > 0$, all of the SHRs of size $2^k + 1$ through 2^{k+1} will at least have the connectivity of the $(2^k + 1)$-SHR. This is due, in part, to the Expansion Lemma (see, e.g., [13]) which states that if G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected. The above allows us to focus on SHRs of size $P = 2^q + 1$, e.g., the integers $0, 1, ..., 2^q$.

We show, by induction on q, that P-SHRs have connectivity of $\lceil \log P \rceil$. For $q = 1$, the 3-SHR is obviously 2-connected. For $q = 2$ and $q = 3$, we can easily show that the 5-SHR is 3-connected and the 9-SHR is 4-connected.

Inductive Hypothesis: Assume that any $(2^q + 1)$-SHR is $q + 1$-connected, i.e., between any two nodes s and t, there exist $q + 1$ node-disjoint paths. Equivalently, if S_1 and S_2 are two disjoint sets of $q+1$ nodes in each, all located within the same $(2^q + 1)$-SHR, then there exist $q + 1$ node-disjoint paths within it that pairwise connect the nodes of both sets.

Let us now examine a $(2^{q+1} + 1)$-SHR. From the inductive hypothesis, for all located within the same $(2^q + 1)$-SHR, there exist $q + 1$ node-disjoint paths between any pair of nodes within the $(2^q + 1)$-SHR. Observe that the degrees of all nodes in the $(2^{q+1} + 1)$-SHR have increased by 1, e.g., the degree of node 0 is now $q + 2$, etc. We must show that for any $0 \leq s < t \leq 2^{q+1}$ there are now $q + 2$ node-disjoint paths between s and t. There are two cases to consider:

(1) If $t - s < 2^q$, then by the Inductive Hypothesis there exist $q + 1$ node-disjoint paths between s and t such that the internal nodes of these paths will be totally contained within a $(2^q + 1)$-SHR whose one endpoint is either node s, or t. The $(q + 2\text{nd})$ path will have one of four possible forms, which will depend on the relative placement of s and t:
$\{s, s-1, s-2, ..., 0, 2^{q+1}, 2^{q+1}-1, 2^{q+1}-2, ..., t+2^q, t\}$, if $s, t \leq 2^q$;
$\{s, s-2^q, s-2^q-1, s-2^q-2, ..., 0, 2^{q+1}, 2^{q+1}-1, 2^{q+1}-2, ..., t\}$, if $s, t \geq 2^q$;
$\{s, s+2^q, s+2^q-1, s+2^q-2, ..., t+1, t\}$, if $s < 2^q < t$;

(2) If $t - s \geq 2^q$, then by the Inductive Hypothesis and the Expansion Lemma there already exist $q + 1$ node-disjoint paths between s and t. Furthermore, all of the internal nodes of these $q + 1$ paths have indices greater than s and less than t. The $(q + 2\text{nd})$ path will have the form: $\{s, s-1, s-2, ..., 0, 2^{q+1}, 2^{q+1}-1, 2^{q+1}-2, ..., t\}$.

□

Before we show that the node-connectivity of an enriched P-SHR is equal to $\lceil \log P \rceil + 1$, we will prove two supporting results in Lemma 1 and Lemma 2. Let r be an integer such that $3 \leq r < P$ and that r is not a power of 2. We enrich a given P-SHR by r. Observe that for any r, $2^q < r < 2^{q+1}$, all of the r-enriched SHRs of sizes $r + 1$ through 2^{q+1} will at least have the connectivity of the $(r + 1)$-SHR. This is again due, in part, to the Expansion Lemma, which states that if G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected. The above allows us to initially focus on showing that the connectivity of r-enriched SHRs of size $P = r + 1$, e.g., the integers $0, 1, ..., 2^q, ..., r$, is $q + 2$. We then show that the connectivity of r-enriched SHRs of size $2^{q+1} + 1$ is equal to $q + 3$.

[Lemma 1] *For any r, $2^q < r < 2^{q+1}$, the connectivity of the r-enriched $(r + 1)$-SHR is equal to $q + 2$.*

Proof. For nodes $0 \leq s < t \leq r$, there are three cases to consider:

(1) If $t - s \geq 2^q$, then by Lemma 1 the connectivity of $SHR_{(s,t)}$ is equal to $q + 1$ (obtained without using the r-connection), where $SHR_{(s,t)}$ means the $(t - s + 1)$-SHR with the set of nodes, $\{s, s+1, ..., t\}$. The $(q+2\text{nd})$ path will have the following form: $\{s, s-1, ..., 0, r, r-1, ..., t\}$.

(2) If $2^{q-1} \leq t - s < 2^q$, then by Lemma 1 there exist q node-disjoint paths between s and t. Furthermore, all of the internal nodes of these q paths have indices greater than s and less than t. The $q+1st$ path will have the following form: $\{s, ..., s_l, s_l + 2^q, ..., t\}$, where node $s_l < s$ is chosen so that the 2^q-connection jump may be executed and

so that $s_l + 2^q \neq r$, unless $t = r$. It is possible that $s_l = s$. The $q + 2nd$ path would be: $\{s, ..., 0, r, ..., t\}$. We can easily force the prefixes $s, ..., s_l$ and $s, ..., 0$ (as well as the suffixes $s_l + 2^q, ..., t$ and $r, ..., t$) to be internally node-disjoint. Note that if $s = 0$, the (q+1st) and (q+2nd) paths will be: $\{0, 2^q, ..., t\}$ and $\{0, r, ..., t\}$ and for $s = 1$, they would be: $\{1, 1 + 2^q, ..., t\}$ and $\{1, 0, 2^q, ..., t\}$, again forcing the suffixes to be internally node-disjoint. The case when $t = r$ would result in paths: $\{s, ..., 0, r\}$ and $\{s, ..., r - 2^q, r\}$.

(3) If $t - s < 2^{q-1}$, then by Lemma 1 there already exist q node-disjoint paths between s and t such that the internal nodes of these paths will be totally contained within a $(2^{q-1} + 1)$-SHR whose one endpoint is either node s, or (mirror image case below) t. The (q + 1st) and (q + 2nd) paths will have one of several possible forms, depending on the relative placement of s and t:

If $t \leqq 2^{q-1}$, then the two additional paths for the subcases of when $s = 0$ or 1, as well as when $t = 2^{q-1}$ or $t = 2^{q-1} - 1$ can be easily constructed outright. In the remaining cases, (i.e., $1 < s < t < 2^{q-1} - 1$), the form of these paths will depend on the positional distance between s and t. If $t - s = 1$, then the paths will be: $\{s, s-1, t\}$ and $\{s, ..., 0, r, ..., t + 2^{q-1}, t\}$. Otherwise, (i.e., $2 \leqq t - s < 2^{q-1}$) we can strategically choose a node within the $\mathrm{SHR}_{(s, s+2^{q-1})}$ that is directly connected to t and which was not used in the construction of the first q paths. The existence of such a node is guaranteed by the fact that $t - s \geqq 2$ and $2 \leqq s < t < 2^{q-1} - 1$. Let us call this free node t_f. In most cases we can allow $t_f = t \pm 1$ or $t \pm 2$. Here, the two additional paths will be: $\{s, ..., 1, 2^q + 1, ..., t + 2^{q-1}, t\}$ and $\{s, ..., 0, 2^q, ..., t_f + 2^{q-1}, t_f, t\}$. Again, we make sure that both prefix and suffix pairs are disjoint.

If $s \geqq 2^{q-1}$, the (q+1st) and (q+2nd) paths will be mirror images of the paths proposed above.

If $s < 2^{q-1} < t$, the form of the (q + 1st) path will be: $\{s, ..., t - 2^{q-1}, t\}$. For the ($q$ + 2nd) path there will be two subcases to consider:

- If $t - s < 2^{q-2}$, the path will be: $\{s, ..., t - 2^{q-2}, t\}$,
- If $t - s \geqq 2^{q-2}$, the path will be: $\{s, ..., t_f - 2^{q-1}, t_f, t\}$, where $0 \leqq t_f - 2^{q-1} < s$, however, for the special case when $t = s + 2^{q-1} - 1$, the path will be: $\{s, ..., 0, 2^q, ..., t + 2, t\}$.

□

[Lemma 2] *For any r, $2^q < r < 2^{q+1}$, the connectivity of an r-enriched $(2^{q+1} + 1)$-SHR is $q + 3$.*

Proof. We must show that for any $0 \leqq s < t \leqq 2^{q+1}$ there are $q + 3$ node-disjoint paths between s and t. Three cases arise:

(1) If $r \leqq t - s$, then by Lemma 1 there already exist $q + 2$ node-disjoint paths between s and t such that the internal nodes of these paths will be totally contained within $\mathrm{SHR}_{(s,t)}$. The (q + 3rd) path will have the form: $\{s, ..., 0, 2^{q+1}, ..., t\}$.

(2) If $2^q \leqq t - s < r$, then by Theorem 1 there already exist $q + 1$ node-disjoint paths between s and t. Furthermore, all of the internal nodes of these $q + 1$ paths have indices greater than s and less than t. The (q + 2nd) and (q+3rd) paths will have the form: $\{s, s-1, ..., s_l, s_l + r, ..., t\}$ and $\{s, ..., 0, 2^{q+1}, ..., t\}$. The forcing of prefixes $s, ..., s_l$ and $s, ..., 0$ (and the suffixes $s_l + r, ..., t$ and $2^{q+1}, ..., t$) to be internally node-disjoint would be handled as before, as would the cases of when $s = 0$ or $t = 2^{q+1}$.

(3) If $t - s < 2^q$, then by Theorem 1 there already exist $q + 1$ node-disjoint paths between s and t such that the internal nodes of these paths will be totally contained within a $(2^q + 1)$-SHR whose one endpoint is either node s, or t. The (q+2nd) and (q+3rd) paths will have one of several possible forms, depending on the relative placement of s and t:

If $t \leqq 2^q$, then the two additional paths for the subcases of when $s = 0$ or 1, or $t = 2^q$ or $2^q - 1$ can, again, be easily constructed outright. In the remaining cases, (i.e., $1 < s < t < 2^q - 1$), the form of these paths will depend on the positional distance between s and t. If $t - s = 1$ then the paths will be: $\{s, s-1, t\}$ and $\{s, ..., 0, 2^{q+1}, ..., t + 2^q, t\}$. Otherwise, (i.e., $2 \leqq t - s < 2^q - 1$) there will again exist at least one node within the $\mathrm{SHR}_{(s, s+2^q)}$ that is directly connected to t and which was not used in the construction of the first $q + 1$ paths. Let us call this node t_f. Now, the two additional paths will be: $\{s, ..., 0, 2^{q+1}, ..., t + 2^q, t\}$ and $\{s, ..., s_l, s_l + r, ..., t_f + 2^q, t_f, t\}$. As usual, we can easily make sure here that both the prefix and suffix pairs are internally node-disjoint.

If $s \geqq 2^q$, the (q + 2nd) and (q + 3rd) paths will be mirror images of the paths proposed above.

If $s < 2^q < t$, the (q + 2nd) path will be: $\{s, ..., t - 2^q, t\}$. For the (q+3rd) path there will be two subcases to consider:

- If $t - s < 2^{q-1}$, then the (q + 3rd) path will be: $\{s, ..., t - 2^{q-1}, t\}$,
- If $t - s \geqq 2^{q-1}$, the (q + 3rd) path will be: $\{s, ..., t_f - 2^q, t_f, t\}$, however, for the special case when $t = s + 2^q - 1$, the path will be: $\{s, ..., 0, 2^{q+1}, ..., t + 2, t\}$.

□

We are now ready to present the main result.

[Theorem 2] *For a fixed r, $3 \leqq r$, which is not a power of 2, the node-connectivity of an r-enriched P-SHR is equal to $\lceil \log P \rceil + 1$, which is also the degree of its first and last nodes.*

Proof. Similar to the proof of Theorem 1, this proof will be by induction. Let r be an integer such that $3 \leqq r < P$ and that r is not a power of 2. We enrich a given P-SHR by

r. It is obvious that the degree of the first and last nodes of this r-enriched P-SHR is $\lceil \log P \rceil + 1$. Observe that for any r, $2^q < r < 2^{q+1}$, all of the r-enriched SHRs of sizes $r+1$ through 2^{q+1} will at least have the connectivity of the $(r+1)$-SHR, which by Lemma 1 is equal to $q+2$. By Lemma 2, the connectivity of r-enriched SHRs of size $2^{q+1}+1$ is equal to $q+3$. This is the basis in the inductive step of our proof, i.e., $i = 1$.

Inductive Hypothesis: Assume that any r-enriched SHR of size $(2^{q+(i)}+1)$ is $q+(i)+2$-connected. We must show that the connectivity of any r-enriched SHR of size $2^{q+(i+1)}+1$ is $q+(i+1)+2$. Let us examine a $(2^{q+(i)+1}+1)$-SHR. As in the proof of Theorem 1, we have to construct an additional path between s and t. Again, there are two cases to consider: $t-s \leq 2^{q+(i)}$ and $t-s > 2^{q+(i)}$. In either case, the construction of the additional path would proceed exactly as in the proof of Theorem 1, with the exception that the inductive hypothesis would assume $q+(i)+2$ paths in the r-enriched $(2^{q+(i)}+1)$-SHR. □

4. Path construction between endpoints for SHRs

For ease of presentation and clarity, let us visualize the problem of SHR (enriched SHR) node-disjoint path construction as a simple (one player) blue/green pebble game, which has only three rules:

.(1) The only pebble movements (hops) allowed are via the existing edges.

(2) A pebble may not be placed on a node previously visited by any other pebble.

(3) If node i is occupied by a pebble, the player may *capture* it by hopping a different color pebble to i and removing both pebbles from the game.

The object of the game is to maximize the number of captured pebbles, given some initial pebble configuration. It should be clear that the traces of the pebble hops uniquely define a set of node-disjoint paths and that a capture indicates a completion of a path between the nodes where the pebbles were originally placed.

Let us now present procedure *CompressSHR* that given t as input generates the $\lceil \log t \rceil$ node-disjoint paths between 0 and t within the $(t+1)$-SHR. We will describe the steps of this procedure in terms of pebble movements.

procedure *CompressSHR(t)*

Initialize: place $\lceil \log t \rceil$ blue and $\lceil \log t \rceil$ green pebbles in locations $\{2^0, 2^1, ..., 2^{\lfloor \log t \rfloor}\}$ and $\{t-2^0, t-2^1, ..., t-2^{\lfloor \log t \rfloor}\}$, respectively.

0. Check for any green and blue pebbles placed on the same node, (including blue on t and green on 0) and if found, capture them.

while any pebbles are left do

1. Identify the green pebble g that is closest and to the right of the rightmost blue pebble b.
2. For all of the remaining green pebbles to the right of b use an appropriate *single-hop* to move them to the left of b. If any land on another blue pebble, consider it a capture.
3. Construct the appropriate sequence of hops for g to capture b.

It is clear that the green pebble movements represent the node t to node s paths and that they are node-disjoint (see Figure 3). Observe that at each loop iteration at least one capture is made, limiting the number of loops to $\lceil \log t \rceil$, making the maximal number of hops made by any pebble to be no more than $\lceil \log t \rceil$. Furthermore, if for some reason (e.g., an incorrect initial placement of one of the blue pebbles) step 1 fails, then let g be the green pebble closest to the rightmost blue pebble and continue with step 3, for that particular while-loop iteration.

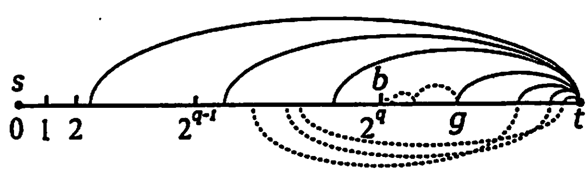

図 3 Compression of an SHR and node-disjoint path construction.

The degree of the first and last nodes in any enriched P-SHR is equal to $\lceil \log P \rceil + 1$. Observe that procedure *CompressSHR*(t) can be naturally extended to include enriched SHRs by initially placing additional blue and green pebbles in locations r and $t-r$, respectively.

[Observation 1] Procedure CompressSHR will capture all of the pebbles in any enriched P-SHR.

Let $s < t$ be two nodes in an N-HR that are $D \leq \lfloor N/2 \rfloor$ positions away from each other. We may construct a (possibly enriched) D-SHR$_{(s,t)}$ that spans s and t by removing from the N-HR all of the nodes (and their edges) outside the line segment (s,t), (i.e., outside $\{s, s+1, ..., t-1, t\}$).

Let us focus on the family of N-HRs that possess the crisscrossing edges connecting nodes of $2^{\lfloor \log N \rfloor}$ positional distance.

[Observation 2] If $D \geqq N - 2^{\lfloor \log N \rfloor}$ then the degree of nodes s and t as well as the number of node-disjoint paths between s and t in the D-SHR$_{(s,t)}$ is equal to $\lceil \log D \rceil + 1$, otherwise it is $\lceil \log D \rceil$.

The next result will allow us to obtain the disjoint-path length bounds between endpoints for SHRs.

[Lemma 3] For any (enriched) D-SHR, the length of the node-disjoint paths constructed by procedure CompressSHR is bounded by $\lceil \log D \rceil + 2$.

These results on the disjoint-paths between endpoints for SHRs and enriched SHRs can be applied to path construct for HRs. Such construction is given in [4].

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