# On the solvability of water distribution networks with unknown pipe characteristics

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*Necessary and sufficient conditions for the solvability of water distribution networks with unknown pipe characteristics are developed. They are predicated on the interdependence between the unknown pipe characteristics and the network hydraulic performance. These characteristics are determined to exactly satisfy defined values of nonlinear boundary equality constraints. The constraint set consists of the stated supply pressure or energy grade requirements to be maintained at critical locations throughout the distribution network. The determination of such conditions is important for a comprehensive and effective modelling and optimization of water networks.*

**Keywords:** network solvability, nonlinear and explicit optimization, water networks, graph theory, linear algebra

## Introduction

There is currently widespread concern regarding the condition of the nation's water supply infrastructure. It is clear that many of the existing water distribution systems will have to be upgraded and modified at great expense if utilities are going to continue to provide reliable systems that cope with sustained urban and rural growth and expansion. These systems are usually complex networks of pipes, pumps, ground-level and elevated storage facilities, wells, treatment plants, and a complement of hydraulic components such as bends, meters, and valves including pressure-regulating, pressure-sustaining, and check valves. Good engineering decisions based on sound analysis procedures will be required if the alterations and improvements to these systems are to be effective and economical. Usual pipe network enhancements, such as addition of new pipes, looping or replacing existing pipes, and addition or upgrading of pumping and storage facilities, require complex and precise hydraulic calculations to be properly evaluated. Current practice involves a tedious trial-and-evaluation procedure that seldom leads to the most effective or most economical solutions for upgrading networks. The complexity of this procedure increases exponentially with the number of proposed modifications and the corresponding system performance requirements. Time constraints or limited understanding or appreciation of pipe network hydraulics often prevent engineers from obtaining hydraulically reliable recommendations. The result of using the trial-and-evaluation approach is often inefficient performance at greater cost.

In recent years, a number of authors have proposed various "implicit" least-cost optimization algorithms for water networks. The algorithms were based on the use of nonlinear programming, dynamic programming, successive linear programming, and general heuristics. An excellent review of the various approaches has been previously provided.[1,2] However, the success of

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these procedures varied, and very few have gained widespread acceptance or have actually been applied to real systems.[1] This is partly because (a) such techniques are generally quite complex involving a considerable amount of mathematical sophistication (e.g., requiring extensive expertise in systems analysis); (b) they are generally subject to oversimplification or significant limitations on pipe network configurations and allowable components; (c) they are generally limited to the optimization of the entire network, whereas most engineering work involves dealing with small portions of existing water distribution systems; (d) they tend to be extremely time consuming (computationally expensive) resulting in added costs and inefficient use of the computer; and (e) they may not lead to a global optimal solution. Engineers are constantly seeking more direct, efficient, and reliable techniques for carrying out these studies. As a result, a robust and efficient, yet numerically "explicit," method is needed.

There is an alternative to applying the trial-and-evaluation and complex implicit mathematical programming procedures to the problem of water distribution parameter calculation. It is possible to explicitly determine proposed system modifications to meet exactly the stated operational specifications. The solution strategy involves adding equations (operating specifications) and corresponding unknowns (pipe characteristics) to the basic set of hydraulic equilibrium equations. The expanded equilibrium equation set can then be reformulated and solved explicitly for the indeterminate pipe characteristics. Such an approach is predicated on relating pipe network characteristics to hydraulic performance in such a manner that the indeterminate characteristics can be selected to meet exactly the target operating specifications, which in most cases will represent the least expensive acceptable system modification for the conditions specified. This will allow practicing engineers more control of the solution process. It allows them to provide sound decision making and to conceive and evaluate efficient and economical recommendations for the improvement and enhancement of existing water distribution networks.

The explicit reformulation of network equations in terms of unknown pipe characteristics has been studied recently by various researchers. Shamir and Howard[3] reformulated the node system of equations to solve for indeterminate pipe resistances. A solution was usually obtained provided a good initial guess was available and the selection of unknowns conformed to certain heuristic rules. Gofman and Rodeh[4] reformulated the loop system of equations in terms of head generators. A head generator was defined as either the positive or negative resistance required to meet the target pressure specified. A positive resistance indicates that the resistance of the pipe should be increased, whereas a negative resistance indicates that a booster pump should be installed. In their paper, Gofman and Rodeh presented necessary and sufficient conditions for the existence of head generators for which their algorithm will converge to a unique solution. Bhave[5] reformulated the Hardy–Cross method of network analysis in terms of unknown pipe resistances. The method was then generalized to systematically, yet iteratively, solve for pipe resistance coefficients and nodal demands.[6] More recently, Bhave[7,8] supplemented his algorithm with a number of heuristic network solvability rules. Finally, Boulos and Wood[9,10] and Boulos and Altman[11] reformulated the full set of equations governing the network hydraulics in terms of unknown pipe characteristics and boundary constraints. In contrast to the previously cited contributions, this full equation approach allows considerably more flexibility in the selection of unknown parameters and constraint specifications in addition to exhibiting superior convergence characteristics. The method was successfully applied to calibration,[12,13] operation,[14] design,[15] and comprehensive modelling[9,10,14] of water distribution networks.

In this paper, the necessary and sufficient conditions for the solvability of water distribution networks with unknown pipe characteristics are developed. The full equation approach of Boulos and Wood[9] and Boulos and Altman[11] is utilized as the basic solution algorithm. It is shown that the network solvability problem depends on the flow distribution and on the manner in which the additional equations and corresponding unknowns are incorporated into the mathematical model of the distribution network. These results will greatly enhance the reliability and integrity of mathematical models of water distribution networks.

## Definition of the network

A distribution network comprises a finite number of oriented unidimensional edges (pipe segments) interconnected by nodes in some specified configuration. Edges may contain pumps (or any other hydraulic component whose differential head versus flow rate characteristic is known, e.g., turbine and heat exchanger) and fittings, such as bends, meters, and valves, where concentrated energy dissipation occurs. Each edge is of defined length, diameter, roughness, and material. The endpoints of each edge are identified as either junction or fixed-grade nodes.

**Definition 1: Junction node.** A junction node is a point of intersecting edges and can also be a point of external consumption where flow can enter or exit the network.

**Definition 2: Fixed-grade node.** A fixed-grade node is a point of known energy grade such as a connection to a well, a treatment plant, a reservoir, an elevated storage facility, or a constant-pressure region.

The network is rooted at any fixed-grade node, which is then referred to as the *datum node*.

## Topological characteristics of networks

A distribution network can be represented by a directed connected graph $G = (N \cup S, E)$ with $n = |N|$ junction nodes, $s = |S|$ fixed-grade nodes, and $e = |E|$ edges. Here, $|U|$ denotes the *cardinality* of the set $U$. $G$

need not be planar, i.e., edges may cross in plan or elevation and yet not join. A graph is *directed* if all of its edges are assigned an orientation, and it is *connected* if it contains a path between any pair of nodes. A *path* is a sequence of edges that allows the movement of flow from one node to another.

The connectedness of $G$ can be verified through the existence of a spanning tree. A *spanning tree* of $G$ with $n + s$ nodes is a set of $n + s - 1$ edges that connects any pair of nodes with a unique path. The spanning tree is rooted at the datum node of $G$. The edges that make up the spanning tree are referred to as *branches*, and the nontree edges are referred to as *links*. A spanning tree is also used to obtain an independent set of circuits. Each circuit is defined as either a fundamental or a pseudocircuit. The basic principle pertaining to the identification of fundamental circuits is that each nontree edge (link) that is added to the spanning tree will uniquely define a closed path with its contiguous tree edges (branches). Such a closed path, which is called a *fundamental circuit*, is independent of any other closed path because it contains an edge (link) that no other fundamental circuit possesses. Because each link will result in the formation of a unique fundamental circuit, it follows that the number of fundamental circuits $l$ (cyclomatic number) of $G$ is

$$l = e - n - s + 1 \tag{1}$$

A *pseudocircuit* is a simple path of connected branches between the datum node and any other fixed-grade note of $G$. A *simple path* is a path in which no node is traversed more than once. It follows that the number of pseudocircuits is $s - 1$. A detailed description of various algorithms for the construction of these circuits has been previously provided.[16]

The connectivity characteristics of $G$ that are needed to formulate the network problem can be described algebraically by topological matrices derived from the structure of the network.[11]

**Definition 3: Junction node incidence matrix.** The junction node incidence matrix $\mathbf{\Lambda}$ of $G$ is an $n$ by $e$ matrix that has a row for every junction node and a column for every edge. For each column, the nonzero entries, $+1$ or $-1$, indicate the junction nodes that begin and end an edge, respectively.

**Definition 4: fp-circuit basis matrix.** The fp-circuit basis matrix $\mathbf{\Gamma}$ of $G$ is an $e - n$ by $e$ matrix that has a row for each circuit (fundamental as well as pseudo) and a column for every edge. In each row, the nonzero entries, $+1$ or $-1$, indicate the presence of directed edges in the circuit that go with or against the circuit orientation, respectively.

**Observation 1.** The rank of $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ are $n$ and $e - n$, respectively; moreover, the row vectors of $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ are mutually orthogonal, that is, $\mathbf{\Lambda\Gamma}^T = \mathbf{0}$. It is also clear that the $e$ by $e$ matrix $\boldsymbol{M} = [{}^{\mathbf{\Lambda}}_{\mathbf{\Gamma}}]$ is nonsingular. For proofs of the above, the reader is referred to Boulos and Altman[11] and Feldmann and Rohrer.[17]

**Definition 5: datum-to-junction node path matrix.** The datum-to-junction node path matrix $\mathbf{\Omega}$ of $G$ is an $n$ by $e$ matrix that has a row for every junction node and a column for every edge. In each row, the nonzero entries, $+1$ or $-1$, indicate the presence of directed edges (branches) in the path from the datum to the specified junction node that go with or against the path orientation, respectively.

**Observation 2.** The rank of $\mathbf{\Omega}$ is $n$. The proof for this observation is given by Fenves and Branin.[18]

When such network components as pressure-regulating and pressure-sustaining valves cause $G$ to be logically separated, i.e., split into $m$ subnetworks each of which satisfies equation (1), then the Euler–Poincare relation will hold

$$e = n + l + s - m \tag{2}$$

**Theorem 1.** Let us consider a sequence of $e_k$ by $e_k$ matrices $\boldsymbol{M}_k = [{}^{\mathbf{\Lambda}_k}_{\mathbf{\Gamma}_k}]$, with $k = 1,\ldots, m$ and $e = \sum_{k=1}^{m} e_k$. Then, the $e$ by $e$ matrix $\boldsymbol{R}$ of the form

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{M}_1 & 0 & 0 & . & 0 \\ 0 & \boldsymbol{M}_2 & 0 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & \boldsymbol{M}_m \end{bmatrix} \tag{3}$$

is nonsingular.

**Proof.** By Observation 1, each submatrix $\boldsymbol{M}_k$ has full row rank ($k = 1,\ldots, m$). It, therefore, suffices to show that the inner product of any two rows, say $i$ and $j$, that do not overlap with the same submatrix, is zero. But, by construction of $\boldsymbol{R}$, it is obvious that such inner product must be zero, which implies that $\boldsymbol{R}$ has full rank. This concludes the proof.

## Mechanical characteristics of networks

Regardless of the topological configuration of $G$, the problem of steady, one-dimensional, incompressible, and isothermal fluid flow in the network is characterized by two physical laws.

**Conservation of mass.** Let $\boldsymbol{Q}$ denote the column vector of volumetric flow rate associated with all the network edges, then the product of each row of $\mathbf{\Lambda}$ with $\boldsymbol{Q}$ must equal the column vector $\boldsymbol{q}$ of external consumptions at the junction nodes (+ if inflow; − if outflow). That is

$$\mathbf{\Lambda}\boldsymbol{Q} = \boldsymbol{q} \tag{4}$$

**Conservation of mechanical energy.** Let $\boldsymbol{\psi}$ denote the column vector of head change across each of the network edges, then the product of each row of $\mathbf{\Gamma}$ with $\boldsymbol{\psi}$ must equal the column vector $\boldsymbol{\phi}$ of head differences between the two boundary circuit nodes (zero for fundamental circuits). That is

$$\mathbf{\Gamma}\boldsymbol{\psi} = \boldsymbol{\phi} \tag{5}$$

for each edge, the head change variable, $\boldsymbol{\psi}$, is a characteristic function describing the relationship between the flow rate and the head difference across that edge. This is a nonlinear function and is given by[19]

$$\boldsymbol{\psi} = \boldsymbol{\zeta}\boldsymbol{Q}^2 + \boldsymbol{\xi}\boldsymbol{Q}^{\sigma} - \boldsymbol{\eta}^2\left(\boldsymbol{\alpha} - \frac{\boldsymbol{\beta}}{\boldsymbol{\eta}^{\nu}}\boldsymbol{Q}^{\nu}\right) \tag{6}$$

where $\zeta$ is the fittings constant given by

$$\zeta = \frac{8K}{g\pi^2 D^4} \tag{7}$$

$\xi$ is the edge resistance constant that may be defined by the Hazen–Williams expression as

$$\xi = \frac{\mu L}{C^v D^b} \tag{8}$$

$K$ is the sum of the minor loss coefficients for the fittings; $g$ is the gravitational acceleration; $L$ is the edge length; $D$ is the edge diameter; $C$ is the Hazen–Williams coefficient of roughness; $\mu$ is a constant that is dependent on the units used; $\alpha$ is the pump shutoff head at zero flow condition; $\beta$ and $\nu$ are the regression coefficient and the exponent of the pump characteristic curve that represents actual pump operation in relation to its reference speed; $\eta$ is the ratio of the pump rotational speed to the pump reference speed; and the exponents $\sigma$ and $b$ are 1.852 and 4.87, respectively. The pump characteristic curve can be obtained from a pump test relating the discharge to the head differential across the pump. The pump operation can also be described by a quadratic characteristic function.[9,11,14]

## Networks with unknown edge characteristics

### *Problem formulation*

Equation (1) and Observation 1 (or equation (2) and theorem 1) provide *necessary* and *sufficient* conditions for the solvability of the network flow problem, respectively. The first condition ensures the assembly of as many equations as there are unknowns (even-determined problem). The second condition secures the framing of an independent set of flow equations. Accordingly, equations (4) and (5) can be combined to solve the network flow distribution problem.[11] The problem is thus reduced to determining a set of $e$ individual flow rates that simultaneously satisfies both equations for specified edge characteristics and network loading (demand distribution) and corresponding operating (e.g., pump heads, tank levels, pressure-regulating valve settings) conditions. The analytical network flow problem can be expressed as

$$F(Q) = \begin{Bmatrix} \Lambda Q - q \\ \Gamma\psi(Q) - \phi \end{Bmatrix} = 0; f_i = 0, \quad i = 1,\ldots,e \tag{9}$$

Edge characteristics can also be treated as unknowns and explicitly determined to meet stated operational pressure (or grade) requirements at designated junction nodes throughout the distribution network. This involves adding equations and corresponding unknowns to the full set of flow continuity and energy equations describing the network hydraulics. The added equations represent the operating pressure conditions specified, whereas the added unknowns represent the edge characteristics to be determined. Because the head change variable, $\psi$, associated with an edge, is a function of the edge resistance constant, fittings constant, length, diameter, roughness, minor loss coefficient, and pump speed, any of these parameters can be selected as an additional variable. Each variable defined requires the introduction of an additional energy equation. The additional equation is written as a pseudocircuit between the datum node and the junction node with specified pressure. Such a junction node is referred to as a *critical* node. This equation can be expressed as

$$\sum_{i=1}^{e}\left[\omega_{j,i}\left(\zeta_i Q_i^2 + \xi_i Q_i^\sigma - \eta_i^2\left(\alpha_i - \frac{\beta_i}{\eta_i^\nu} Q_i^\nu\right)\right)\right] = \phi_j' \tag{10}$$

where $\omega_{j,i}$ designates the $j$th row and $i$th column of $\Omega$; and $\phi_j'$ is the energy grade difference between critical node $j$ and the datum node. Up to $n$ distinct junction nodes may have their pressure specified. Each pressure condition imposed uniquely defines the characteristic of an edge. However, only one characteristic per edge may be treated as an unknown. Because each pressure condition that is added uniquely specifies an extra unknown edge characteristic, it follows that equation (1) can be reformulated as

$$e + p = n + l + s + c - 1 \tag{11}$$

showing that $p$ must equal $c$, where $p$ and $c$ are the number of unknown edge characteristics and specified pressures, respectively. The resulting even-determined analytical network problem can be expressed as

$$\Lambda Q = q \tag{12a}$$

$$\Gamma\psi(Q,\Theta) = \phi \tag{12b}$$

$$\Omega\psi(Q,\Theta) = \phi' \tag{12c}$$

or in a more compact form

$$F(Q,\Theta) = 0; f_i = 0, i = 1,\ldots,e + p \tag{13}$$

where $\Theta$ is a $p$ dimensional vector of unknown edge characteristics, with $p \leq n$. The resulting formulation constitutes a quasilinear boundary value problem, whose solution gives the network flow distribution, $Q$, and the $p$ additional edge characteristics, $\Theta$.

### *Solution procedure*

The simultaneous solution of $e + p$ quasilinear algebraic equations is required to determine the flow rate in each edge and the designated unknown edge characteristics. This solution can be obtained iteratively, using the incremental Newton–Raphson method.[9,11,16] For $\nu = 0$ onward, the iterative procedure can be expressed as

$$F'(Q,\Theta)^{(\nu)}\begin{Bmatrix} \Delta Q \\ \Delta\Theta \end{Bmatrix}^{(\nu+1)} = -F(Q,\Theta)^{(\nu)} \tag{14a}$$

with

$$Q^{(\nu+1)} = Q^{(\nu)} + \Delta Q^{(\nu+1)} \tag{14b}$$

and

$$\Theta^{(\nu+1)} = \Theta^{(\nu)} + \Delta\Theta^{(\nu+1)} \tag{14c}$$

where $\Delta Q$ and $\Delta\Theta$ are the incremental flow rate and unknown edge characteristics vectors, respectively. The superscript in parentheses designates the Newton's iteration number, with $\nu$ being the current Newton's step. The Jacobian matrix $F'$, which is defined as the first partial differential of $F$ with respect to $Q$ and $\Theta$, is given by

$$F'(Q,\Theta) = \begin{bmatrix} \frac{\partial f_1}{\partial Q_1} & \cdots & \frac{\partial f_1}{\partial Q_e} & \frac{\partial f_1}{\partial \Theta_1} & \cdots & \frac{\partial f_1}{\partial \Theta_p} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \frac{\partial f_{e+p}}{\partial Q_1} & \cdots & \frac{\partial f_{e+p}}{\partial Q_e} & \frac{\partial f_{e+p}}{\partial \Theta_1} & \cdots & \frac{\partial f_{e+p}}{\partial \Theta_p} \end{bmatrix} \tag{15}$$

with its coefficients being explicitly determined at each iteration. Analytical expressions for these terms can be easily derived. The iterations continue until the relative change in $Q$ and $\Theta$ between two successive iterates is less than some specified tolerance.

*Problem solvability*

The problem of solvability of distribution networks with unknown edge characteristics is dependent on the network flow distribution and on the manner in which the unknown edge characteristics and corresponding critical junction nodes are topologically allocated.

Unsolvable situations due to the network flow distribution can occur for "any" explicit or optimization algorithm and essentially occur because the indeterminate edge characteristics selected are unable to control the pressure specifications for the baseline conditions. For example, the setting of a throttle valve in an edge leading from a storage tank cannot affect the pressure at a critical junction node where none of the supplied flow originates from that storage tank. Similarly, an edge with an unknown diameter may be closed (valve shut) and the pressure specification exceeded. Therefore, no solution exists for the diameter that will meet the pressure imposed. These situations can be avoided by ensuring the existence of a simple path of flow-directed edges between the critical node with targeted pressure and the edge with the indeterminate characteristic, inclusive. Note that an edge with a zero flow is not flow-directed.

Unsolvable situations due to the topological variable/constraint allocation occur because the indeterminate edge characteristics chosen are positioned such that they cannot independently or uniquely control the pressure set at the critical nodes. For example, it is not possible to determine the diameter and roughness of a particular edge to meet two pressure constraints anywhere in the network. To prevent such improper topological allocations using the full equation set approach, sufficient network-solvability conditions are developed. These are

1. **Condition 1.** $G$ must be acyclic, i.e., no flow circulation should exist around any fundamental circuit.
2. **Condition 2.** Each pseudocircuit that is added must contain at least one edge with an unknown characteristic and uniquely correspond to a specific row of $\Omega$. In addition, each such pseudocircuit must uniquely contain an edge with an unknown characteristic that no other pseudocircuit in $\Omega$ possesses, i.e., no two pseudocircuits can have the same new unknown.

The proof that these conditions are indeed sufficient follows from the framing of the algebraic topological composition of the coefficient matrix of equation (14a) and is given below. Note that for each additional unknown associated with an edge $j$ a column of zeroes will be appended to $\Lambda$, while the additional column of $\Gamma$ and $\Omega$ will be identical to their respective $j$th columns.

**Observation 3.** If $T$ is a set of edges corresponding to any spanning tree over the junction nodes of $G$, then there exists a labeling of the edges of $G$ such that $\Omega$ may be partitioned into $\Omega_1$ and $\Omega_2$, where $\Omega_1$ is an $n$ by $n$ full rank matrix and $\Omega_2$ is an $n$ by $e - n$ zero matrix.

**Proof.** Simply relabel $G$ so that the datum-node-to-spanning-tree edge and the $(n - 1)$ edges of the spanning tree are assigned labels $1$ to $n$.

Also, observe that if some $\Omega^p$ has only $p \leq n$ rows, then the ranks of $\Omega^p$ and $\Omega_1^p$ are both $p$. Let $\Lambda = [\Lambda_1\ \Lambda_2]$ and $\Gamma = [\Gamma_1\ \Gamma_2]$, where $\Lambda_1$ and $\Gamma_1$ are $n$ by $n$ and $e - n$ by $n$ matrices and $\Lambda_2$ and $\Gamma_2$ are $n$ by $e - n$ and $e - n$ by $e - n$ matrices, respectively. It follows that

$$M = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Gamma_1 & \Gamma_2 \end{bmatrix} \tag{16}$$

Note that since $G$ is acyclic (by Condition 1), there will always exist an ordering of the edges in $T$ such that $\Lambda_1$ is a lower triangular matrix with $\pm 1$'s on its diagonal.

**Observation 4.** The nontree edges of $G$ may be labeled so that $\Gamma_2$ is a diagonal matrix with $\pm 1$'s on the diagonal.

**Proof.** Simply label each nontree edge associated with the $i$th row of $\Gamma_2$ with the label $n + i$.

Let $M^p$ designate the algebraic topological composition of the expanded Jacobian matrix, constructed following the aforementioned network solvability rules, to solve for up to $p \leq n$ additional unknowns. Then, $M^p$ can be put in the following form

$$M^p = \begin{bmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3^p \\ \Gamma_1 & \Gamma_2 & \Gamma_3^p \\ \Omega_1^p & \Omega_2^p & \Omega_3^p \end{bmatrix} \tag{17}$$

where $\mathbf{\Omega}^p = [\mathbf{\Omega}_1^p\ \mathbf{\Omega}_2^p]$, $\mathbf{\Omega}_1^p$ is a $p$ by $n$ matrix, $\mathbf{\Omega}_2^p$ is a $p$ by $e - n$ matrix, and $\mathbf{\Omega}_3^p$ is a nonsingular (by Observation 2 and Condition 2) $p$ by $p$ matrix ($1 \leq p \leq n$). Also, $\mathbf{\Lambda}_3^p$ and $\mathbf{\Gamma}_3^p$ are $n$ by $p$ and $e - n$ by $p$ matrices, respectively. By construction, all of the entries in $\mathbf{\Lambda}_3^p$ and $\mathbf{\Omega}_2^p$ are zero. An interesting observation is that if $p = n$, then $\mathbf{\Gamma}_1 = \mathbf{\Gamma}_3^n$ and $\mathbf{\Omega}_1 = \mathbf{\Omega}_3^n$ (modulo some permutation of the last $n$ columns of $\boldsymbol{M}^n$).

**Theorem 2.** The matrix $\boldsymbol{M}^n$ is nonsingular.

**Proof.** Because $\mathbf{\Omega}_1^n$ is nonsingular, it can be used to zero out the entries of $\mathbf{\Gamma}_1$. Note that the matrix $\mathbf{\Gamma}_3^n$ will also be zeroed out by this operation. $\mathbf{\Gamma}_2$, on the other hand, will remain unchanged, because $\mathbf{\Omega}_2^n$ is, by construction, an $n$ by $e - n$ zero matrix. Because $\mathbf{\Omega}_3^n$ is nonsingular, $\mathbf{\Omega}_1^n$ and $\mathbf{\Omega}_3^n$ can be put into a lower triangular form $\boldsymbol{T}_\Omega^n$ (with $\pm 1$'s on the diagonal). The resulting matrix will have the form shown below

$$\begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 & 0 \\ \mathbf{0} & \mathbf{\Gamma}_2 & 0 \\ \boldsymbol{T}_\Omega^n & \mathbf{0} & \boldsymbol{T}_\Omega^n \end{bmatrix} \tag{18}$$

Now, the diagonal matrix $\mathbf{\Gamma}_2$ can be used to zero out $\mathbf{\Lambda}_2$. The resulting matrix is lower triangular with nonzero diagonal entries and, hence, nonsingular. This concludes the proof.

**Corollary 1.** The matrix $\boldsymbol{M}^p$, $1 \leq p \leq n$, is nonsingular.

Equation (11) and Corollary 1 provide the *necessary* and *sufficient* conditions, respectively, for the existence of edge characteristics, for which equation (14a) is solvable.

## Illustrative example

As an illustrative example, we consider the water distribution network shown in *Figure 1*. A labeling scheme is shown for edges and nodes. As can be seen from the figure, this network contains six directed edges, four junction nodes and one fixed-grade node. This fixed-grade node corresponds to a reservoir with a water level of 50 m. The fixed-grade node is identified as the datum node, and a possible spanning tree will consist of edges (branches) 1, 3, 4, and 5. The two remaining nontree edges (links) uniquely define two fundamental circuits. Link 2 defines a fundamental circuit consisting of edges 2, 3, and 4, whereas link 6 defines a fundamental circuit comprising edges 4, 5, and 6. Note that equation 1 is satisfied. The pertinent edge and junction node characteristics are provided in *Tables 1* and *2*, respectively. Minor loss coefficients for the edges are neglected. The characteristic data associated with the pump in edge 1 are given in *Table 3*. The mathematical model can be formulated as follows

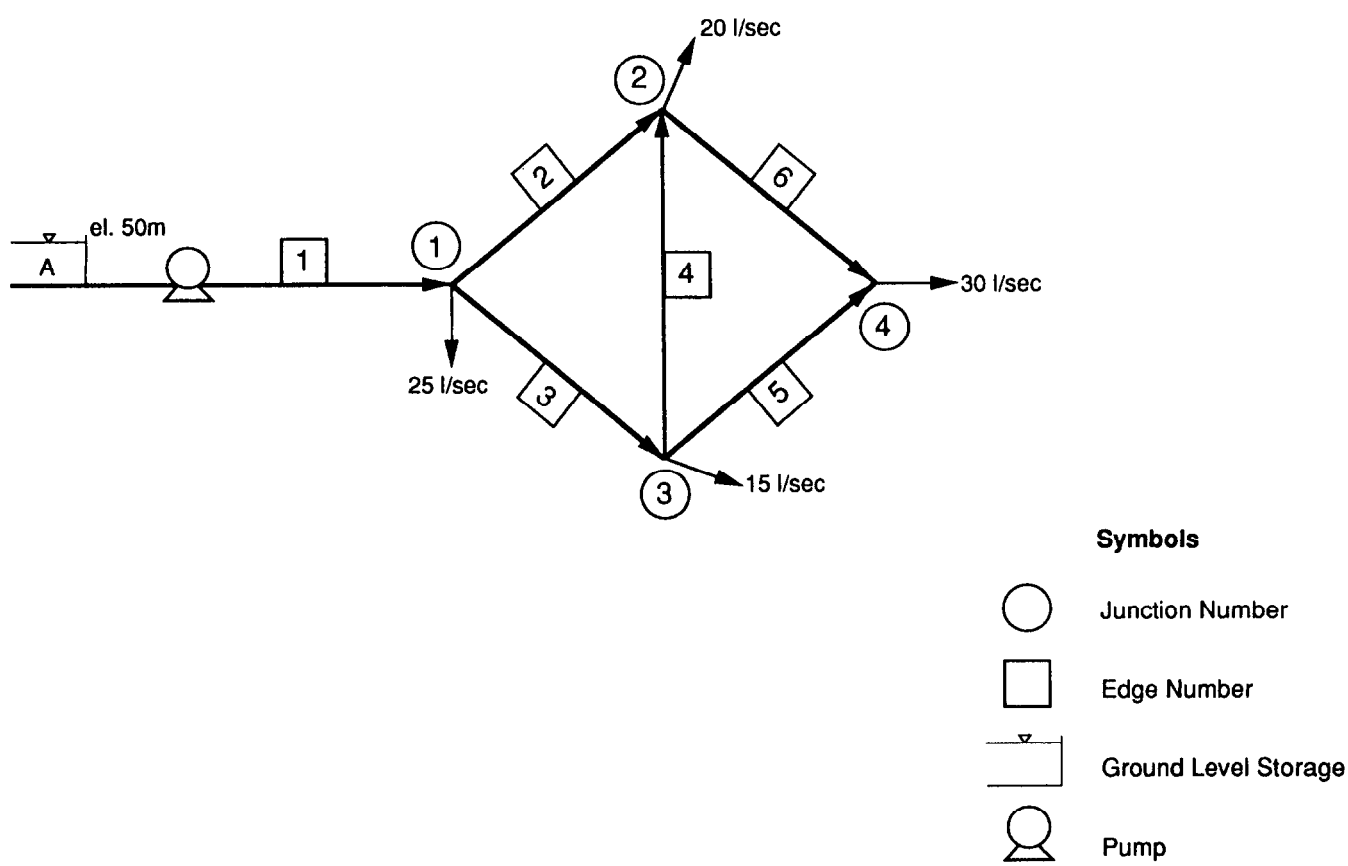


**Figure 1.** Example of network.

**Table 1.** Edge characteristics.

| Edge number | Head node | Tail node | Length (m) | Diameter (mm) | Roughness coefficient | Flow rate (l/sec) |
|---|---|---|---|---|---|---|
| 1 | A | 1 | 300.0 | 400.0 | 130.0 | 90.00 |
| 2 | 1 | 2 | 280.0 | 350.0 | 120.0 | 32.71 |
| 3 | 1 | 3 | 280.0 | 350.0 | 120.0 | 32.29 |
| 4 | 3 | 2 | 400.0 | 300.0 | 110.0 | 2.18 |
| 5 | 3 | 4 | 300.0 | 280.0 | 110.0 | 15.11 |
| 6 | 2 | 4 | 300.0 | 280.0 | 110.0 | 14.89 |

**Table 2.** Junction node characteristics.

| Node number | Elevation (m) | Demand (l/sec) | Pressure (kPa) |
|---|---|---|---|
| 1 | 90.0 | 25.0 | 348.18 |
| 2 | 85.0 | 20.0 | 395.99 |
| 3 | 80.0 | 15.0 | 445.05 |
| 4 | 75.0 | 30.0 | 493.00 |

**Table 3.** Pump head-flow data.

| Head (m) | Flow rate (l/sec) |
|---|---|
| 100.0 | 0.0 |
| 80.0 | 80.0 |
| 40.0 | 160.0 |

*Continuity*

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} -25 \\ -20 \\ -15 \\ -30 \end{Bmatrix} \quad (19)$$

*Energy*

$$\begin{bmatrix} 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (20)$$

The solution of this quasilinear system by the Newton method gives the flow distribution shown in *Table 1*. The resulting junction node pressures are presented in *Table 2*.

It is desired to explicitly calculate the decrease in the pump rotational speed that is required to lower the pressure at junction node 4 to 300 kPa (grade of 105.59 m). For this case, junction node 4 is designated as a critical node and $\eta$ is designated as a decision variable for direct calculation. Note that equation (11) is satisfied (i.e., $p = c = 1$). The target pressure specified defines an additional energy equation between the datum node and node 4. This is

$$[1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{Bmatrix} = \{-55.59\} \quad (21)$$

Note that the head change variable $\psi_1$ is now a function of both $Q_1$ and $\eta$ and that Conditions 1 and 2 are satisfied. The simultaneous solution of these seven quasilinear algebraic equations was obtained in three Newton iterations. The convergence tolerance was set to 0.001. The results are summarized in *Table 4* and show that a pump speed ratio of 0.89 is required.

## Conclusion

The necessary and sufficient conditions for the solvability of mathematical models of water distribution networks with unknown pipe characteristics have been presented. Because of its inherent simplicity, flexibility, fast convergence, and breadth of application, the full equation set approach governing the network hydraulics was used as the basic solution procedure. The method is general and applicable to any liquid distribution network. The conditions presented provide a powerful tool for enhancing the reliability and integrity of mathematical models of water distribution networks.

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**Table 4.** Computational results.

| Edge number | Flow rate (l/sec) | Pump speed ratio | Node number | Pressure (kPa) |
|---|---|---|---|---|
| 1 | 90.00 | 0.89 | 1 | 155.18 |
| 2 | 32.71 | — | 2 | 202.99 |
| 3 | 32.29 | — | 3 | 252.06 |
| 4 | 2.18 | — | 4 | 300.00 |
| 5 | 15.11 | — | | |
| 6 | 14.89 | — | | |

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