Solving Flow-Constrained Networks: Inverse Problem

By Tom Altman[1] and Paul F. Boulos,[2] Member, ASCE

Abstract: Necessary and sufficient conditions for solving flow-constrained networks are developed. These conditions are predicated on the interrelation between the decision variables specified and the network-flow hydraulics. They are applicable to even-determined models of water-distribution systems. The decision variables are selected from a wide range of pipe-system parameters and are calculated explicitly to satisfy stated flow-equality constraints. A continuous-variable space is assumed for the solution. The determination of such conditions is important for comprehensive and effective modeling and optimization of water-distribution networks. These conditions can serve as guidelines to supplement existing procedures of network analysis.

INTRODUCTION

Many methods now exist for water-distribution network analysis. They are predicated on the use of laws of mass conservation and fluid mechanics. The capabilities of these methods are characterized by the size of the variable domain (i.e., the number of indeterminate parameters that can be solved for) and the boundary-constraint set; the larger the better. These methods belong to two types of network-analysis problems with increasing capabilities: the forward and inverse problems. In the forward problem, the system's hydraulic behavior (i.e., the flow in each pipe, the pressure at each node, and the operating heads for pumps) is determined for specified pipe-system characteristics as well as demand loading and operating conditions. In the inverse problem, selected pipe-system characteristics are treated as variables and are determined to meet designated flow and/or pressure specifications. This paper addresses a version of the latter type.

Inverse problems have been effectively used in many different types of application of distribution-system analysis, including design (Ormsbee and Wood 1986a; Boulos and Wood 1991a), operation (Boulos and Wood 1991b), calibration (Ormsbee and Wood 1986b; Bhave 1988; Boulos and Ormsbee 1991a,b), leakage characterization (Sterling and Bargiela 1984; Jowitt and Xu 1990; Pudar and Liggett 1992; Liggett 1993), and comprehensive modeling (Bhave 1986; Boulos and Wood 1990a,b; Boulos and Altman 1991). Inverse problems of even-determined models are always successful in securing the solution sought for a mathematically well-posed model (Boulos et al. 1993). Inverse problems are even-determined when the number of indeterminate parameters equals the number of boundary specifications imposed. The problem of solvability of these models is dependent on the network-flow distribution and on the manner in which the decision parameters and corresponding boundary specifications are topologically allocated (Gofman and Rodeh 1981; Boulos and Wood 1990b, 1992; Boulos et al. 1993). Analytical procedures have already been developed by Gofman and Rodeh (1981) and Boulos et al. (1993). Shamir and Howard (1968, 1970, 1977) and Bhave (1990, 1991) proposed heuristic rules for model solvability. The previous rules were restricted to solving acyclic networks (i.e., no flow circulation exists around any of the network loops) in the presence of pressure constraints only. Flow constraints were not explicitly addressed, on the assumption that a solution existed and was obtainable.

Flow-constrained models may be useful in many different applications of network analysis. Specifying the flow into or out of a storage facility; the flow through a flow limiting device (i.e., flow-control valve); the pump discharge flow; and the outgoing flow from a well, a reservoir, or a treatment plant are only a few examples of such applications. However, the mathematical formulation of such models is not easily characterized. As a result, the determination of the solvability conditions for these models would be highly desirable.

In this paper, the necessary and sufficient conditions for the solvability of the inverse problem with respect to flow constraints are developed. These are derived analytically from a combination of graph-theory and linear-algebra results. The inverse problem is formulated in terms of unknown pipe characteristics and is developed from a reformulation of the basic steady-state network-flow equations. The constraint set is expressed in terms of stated volumetric flow-rate requirements to be maintained in designated pipes. It is assumed that the network topology along with the location of the various network components are fixed. The determination of such conditions is important for a comprehensive and effective modeling and optimization of water-distribution networks. These conditions can serve as guidelines to supplement existing procedures of network analysis. It is shown that special care is needed in using flow-constrained models of network analysis.

MATHEMATICAL FORMULATION

The network-flow model is formulated concisely, so that the derivation of the solvability conditions can be done with the notations, definitions, and equations presented here.

Network Definition

A distribution network may be represented by a directed connected graph G comprising a finite number of edges (pipes) connected to one another by nodes. Edges may contain pumps or any hydraulic device with a different type of head-flow-rate relationship. At the end of each edge are nodes with known energy grade (fixed-grade nodes) or external water consumption (junction nodes). Water flows through the edges and can enter or exit the graph at any node. The graph is rooted at any fixed-grade node, which is then referred to as the datum node.

Network Topology

The network graph may be constructed from topological consideration. Starting at the datum node, a subgraph of G that spans the datum and all the junction nodes may be defined. Such a subgraph is referred to here as a junction-spanning tree (JST). It is characterized by the property that inflow to any junction node can occur only through a single edge,

[1]Assoc. Prof., Dept. of Computer Sci. and Engrg., Univ. of Colorado at Denver, 1200 Larimer Street, Denver, CO 80217.

[2]Dir., Water Distribution Technol., Montgomery Watson, 300 North Lake Avenue, Ste. 1200, Pasadena, CA 91101.

Note. Discussion open until October 1, 1995. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this technical note was submitted for review and possible publication on April 19, 1994. This technical note is part of the ***Journal of Hydraulic Engineering***, Vol. 121, No. 5, May, 1995. ©ASCE, ISSN 0733-9429/95/0005-0427–0431/$2.00 + $.25 per page. Technical Note No. 8362.

J. Hydraul. Eng. 1995.121:427-431.

whereas outflow may occur by means of several edges. All the edges that define the JST are referred to as branches, and the non-JST edges are referred to as links. There usually exist more than one JST for a given graph. Each link that is added to the JST will uniquely create one loop. Network loops may be of two types: fundamental and pseudo. A fundamental loop comprises only those edges connecting junction nodes, and a pseudoloop must contain exactly two edges that connect the datum node and some fixed-grade node to the remaining JST edges. Due to water-quality (so as to avoid stagnation of water in dead-end mains) and reliability (so as to decrease network vulnerability caused by edge failures) considerations, water-distribution networks usually contain fundamental loops. Fig. 1 shows a directed network graph within which a subset of branches (shown solids) defines a JST. The broken lines represent links forming loops. In addition to the JST (e_1, e_2, e_5, e_6) depicted in Fig. 1, other possible JSTs comprise the following combinations of edges (e_1, e_4, e_5, e_6), (e_1, e_2, e_4, e_6), (e_3, e_4, e_5, e_6), (e_2, e_3, e_5, e_6), and (e_2, e_3, e_4, e_6). For a connected graph containing e edges, n junction nodes, and s fixed-grade nodes, the number of pseudoloops is $s - 1$, and the number of fundamental loops l is defined by the following Euler relation:

$$l = e - n - s + 1 \tag{1}$$

Steady-State Network Equations

There are two main physical laws governing the equilibrium behavior of the system: Kirchhoff's node and loop laws (Osiadacz 1987).

Kirchhoff's Node Law: This states that the algebraic sum of the flows at any junction node is zero. This implies that the external water demand at any junction node is equal to the sum of its incoming and outgoing edge flows. This law may be expressed in the following matrix form:

$$\mathbf{\Lambda Q} + \mathbf{q} = \mathbf{0} \tag{2}$$

where $\mathbf{\Lambda}$ = edge-junction node incidence (n by e) matrix; Q = column vector of edge flow rate; and $\mathbf{q}$ = column vector of external nodal demand (negative if inflow). The nonzero coefficients of the matrix are -1 or $+1$ and correspond to incoming or outgoing edges, respectively. For the network shown in Fig. 1, the junction node equations are as follows:

$$\begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} + \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3}$$

Kirchhoff's Loop Law: This states that the algebraic sum of headloss (or gain) along each loop is zero. This law may be expressed in the following matrix form:

$$\mathbf{\Gamma\Psi} + \mathbf{H} = \mathbf{0} \tag{4}$$

where $\mathbf{\Gamma}$ = edge-loop incidence ($l + s - 1$ by e) matrix; $\mathbf{\Psi}$ = column vector of headloss (or gain) associated with each edge; and $\mathbf{H}$ = column vector of piezometric-head difference between the two fixed-grade nodes in each pseudoloop (zero for fundamental loops). The nonzero coefficients of the matrix are -1 or $+1$ and denote the edge orientation that goes with or against the loop circulation, respectively. For the example network, the loop equations are

$$\begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{Bmatrix} + \begin{Bmatrix} 0 \\ H_6 - H_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{5}$$

The headloss variable, ψ, associated with an edge, is a nonlinear characteristic function of the edge flow rate and may be expressed in the following form:

$$\psi = \phi Q^{\sigma} \tag{6}$$

where the function ϕ and its parameters and the exponent σ can vary depending on the approximating flow-resistance law selected and the type of hydraulic device (e.g., pump, valve) being modeled.

Eqs. (2) and (4) may be paired to define the analytical flow-distribution problem as follows (Boulos and Altman 1991, 1993):

$$\mathbf{F(Q)} = \begin{Bmatrix} \mathbf{\Lambda Q} + \mathbf{q} \\ \mathbf{\Gamma\Psi} + \mathbf{H} \end{Bmatrix} = 0; \quad f_i = 0, i = 1, \ldots, e \tag{7a,b}$$

whose solution gives the edge flow rates. This characterizes the forward problem. The simultaneous solution of this system of quasi-linear algebraic equations may be obtained iteratively using the Newton-Raphson method. The iterations continue until the relative change in flow rates between two successive iterates is less than a specified tolerance. The piezometric head at each junction node can then be computed by starting at the datum node and proceeding into the network while adding or subtracting head changes based on the solution for edge flow rates.

Inverse Problem

The forward problem determines the hydraulic state of the system under a given set of demand loading and operating conditions. A principal application of the forward problem is to evaluate the hydraulic performance of the distribution system based on selected schemes for many types of network analysis (e.g., design, operation, calibration). This is essentially a trial-and-error procedure in which selected system parameters are repetitively adjusted until the system-performance criteria targeted are eventually satisfied. An alternative approach is the use of the inverse problem. Differing from the forward problem, the inverse problem allows for the direct determination of selected physical parameters that meet the stated system-performance specifications. Only edge characteristics are considered here. The mathematical formulation of the inverse problem is described in the following.

The inverse problem is derived in a straightforward manner from an extension of the basic network-flow model. The objective is to simultaneously satisfy the steady-state network-equation set and the targeted flow constraints. This approach requires that the expanded network equations [(7)] be augmented with an additional equation for each additional un-

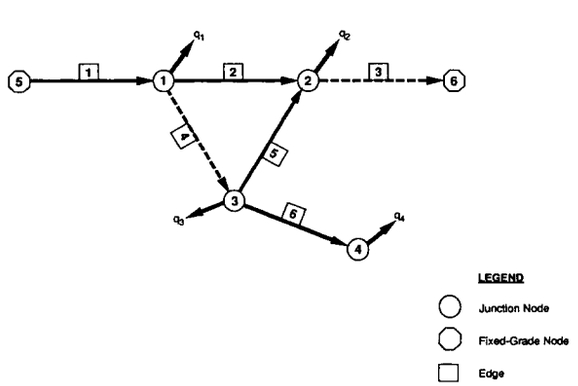

FIG. 1. Example Distribution Network

J. Hydraul. Eng. 1995.121:427-431.

known. The added equation represent the stated volumetric flow (velocity) specifications, whereas the added unknown represents some system parameter (other than flow rate) to be determined. Each additional equation is written as a junction-node equation for a fictitious junction node of indegree 1 connecting the corresponding flow-constrained edge. Such an edge will be referred to as a critical edge. This equation represents an equality constraint that may be expressed as follows: (Boulos and Altman 1991):

$$Q_i = q_{c_i} \tag{8}$$

where q_{c_i} = set flow in edge i. The indegree of a node is defined as the number of incoming edges incident to that node. The indeterminate parameter may be chosen from a wide range of edge characteristics (e.g., diameter, roughness, valve setting). This parameter can then be determined in order to satisfy the flow condition imposed by the additional nodal equation. For example, an extra nodal equation may be specified to define the discharge flow rate of a throttle valve in an edge leading from a treatment-plant facility. Such a condition may be used to specify both the flow rate and piezometric head at the plant. The augmented system of (e + 1) equations can then be explicitly solved for both the flow rates and the minor loss coefficient for the valve (and thereby the valve setting) that produces the designated edge flow rate. Similarly, additional nodal equations may be specified, thus introducing additional indeterminate edge parameters. Because each flow condition that is added uniquely specifies an extra unknown parameter, it follows that (1) may be reformulated as follows:

$$e + p = n + c + l + s - 1 \tag{9}$$

asserting that p must equal c, where p and c = number of unknown parameters and flow constraints, respectively. This equation represents the necessary condition for the solvability of the inverse problem by ensuring the assembly of as many equations as there are unknowns. The resulting even-determined mathematical model may be expressed as follows:

$$\mathbf{A}\mathbf{Q} + \mathbf{q} = 0; \quad \mathbf{\Gamma}\mathbf{\Psi} + \mathbf{H} = 0; \quad \mathbf{\Phi}\mathbf{Q} - \mathbf{q}_c = 0 \tag{10a–c}$$

or in a more compact form

$$\mathbf{F}(\mathbf{Q}, \boldsymbol{\theta}) = 0; \quad f_i = 0, i = 1, \ldots, e + p \tag{11}$$

in which $\mathbf{\Phi}$ = a c by e Boolean matrix in which the nonzero column entries correspond to the critical edges; and $\boldsymbol{\theta}$ = vector of unknown parameters. This model may be solved iteratively using the Newton-Raphson method, whose solution gives the edge flow-rate vector, $\mathbf{Q}$, and the p additional parameters, $\boldsymbol{\theta}$. This characterizes the inverse problem of flow-constrained network models.

Problem Solvability

For the simple case in which $\boldsymbol{\theta}$ is zero ($p = 0$), the inverse approach is reduced to solving the basic network-flow model [(7)] and a convergent solution is virtually assured (Boulos and Altman 1991, 1993). This is not always the case for flow-constrained networks. The solvability of these models is dependent on the network flow distribution and on the manner in which the unknown parameters and corresponding critical edges are topologically allocated.

Unsolvable situations due to the network flow distribution essentially occur because the indeterminate parameters selected are unable to control the flow specifications for the baseline conditions. For example, the setting of a throttle valve in an edge leading from a treatment plant cannot affect the flow in a critical edge where none of the supplied flow emanates from that plant. Similarly, the size of an edge leading from a treatment facility cannot affect the outgoing flow from the plant (assuming that the treatment plant is the only fixed-grade node in the system). Continuity consideration dictates the edge flow rate value. It is also not possible to determine the diameter for that edge that will meet a specified flow anywhere in the system. These situations are difficult to anticipate and require a thorough examination of the interaction between the selected parameters and the network hydraulic performance under the baseline conditions.

Unsolvable situations due to the topological parameter/constraint allocation occur because the indeterminate parameters and critical edges chosen are positioned such that (10a)–(10c) are linearly dependent. The following example depicts this situation. Consider the case in which all edges impinging on a particular junction node are flow-constrained. The nodal equation for that junction node and the added flow equality constraints will be linearly dependent. To prevent such improper topological allocations, the sufficient condition for network solvability is developed. The sufficient condition is: only non-JST edges (links) can be flow-constrained. Furthermore, each critical (link) edge must uniquely provide a single indeterminate parameter. This condition restricts the number of flow constraints to be, at most, equal to the number of loops in the system (i.e., $p = c \leq e - n$). The proof that this condition is indeed sufficient follows from the framing of the algebraic topological composition of the coefficient matrix of (11), and is given in Appendix I.

NUMERICAL RESULTS

Our running example is used here. Tables 1 and 2 summarize the pertinent edge and junction node characteristics, respectively. SI units and the Hazen-Williams headloss equation are employed for this example. Nodes 5 and 6 are fixed-grade nodes with piezometric heads of 250 m and 200 m, respectively. The headloss variable ψ is given by the following

$$\psi = \frac{10.69L}{C^{1.852}D^{4.87}}\mathbf{Q}^{1.852} \tag{12}$$

where L = edge length (m); C = Hazen-Williams coefficient of roughness; and D = edge diameter (m). The mathematical model of the forward problem can be formulated as follows:

Nodal equations

$$\begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} + \begin{Bmatrix} 0.06 \\ 0.08 \\ 0.02 \\ 0.04 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{13}$$

Loop equations

$$\begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -50 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14}$$

The solution of this quasi-linear system by the Newton-Raphson method gives the flow distribution shown in Table 1. The resulting piezometric heads at the junction nodes are presented in Table 2.

To illustrate the inverse problem formulation, a hypothetical case follows. It is desired to explicitly calculate the diameters of edges 3 and 4 to meet targeted flow-rate values of 600 L/s (0.6 m³/s) and 200 L/s (0.2 m³/s) in these edges,

J. Hydraul. Eng. 1995.121:427-431.

TABLE 1. Edge Characteristics

Edge number (1)	Head node (2)	Tail node (3)	Length (m) (4)	Diameter (mm) (5)	Roughness coefficient (6)	Flow rate (L/s) (7)	Velocity (m/s) (8)
1	5	1	1,000.0	800.0	120.0	1,008.49	2.01
2	1	2	1,000.0	400.0	120.0	544.81	4.34
3	2	6	1,000.0	800.0	120.0	808.49	1.61
4	1	3	1,000.0	400.0	120.0	403.69	3.21
5	3	2	1,000.0	400.0	120.0	343.69	2.73
6	3	4	1,000.0	400.0	120.0	40.00	0.32

TABLE 2. Junction-Node Characteristics

Junction number (1)	Demand (L/s) (2)	Piezometric head (m) (3)
1	60.0	245.46
2	80.0	203.02
3	20.0	221.10
4	40.0	220.76

TABLE 3. Computational Results

Edge number (1)	Flow rate (L/s) (2)	Velocity (m/s) (3)	Diameter (mm) (4)	Junction number (5)	Piezometric head (m) (6)
1	800.00	1.59	—	1	247.04
2	540.00	4.30	—	2	202.33
3	600.00	1.89	636.4	3	161.05
4	200.00	3.27	279.0	4	119.43
5	140.00	1.11	—	—	—
6	40.00	0.32	—	—	—

respectively. Note that the necessary ($p = c = 2$) and the sufficient (edges 3 and 4 are non-JST links) conditions are satisfied. This inverse problem requires the addition of two nodal equations, which can be expressed as follows:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} - \begin{Bmatrix} 0.6 \\ 0.2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{15}$$

The resulting even-determined model is described by eight quasi-linear algebraic equations with eight unknowns (six flow rates and two diameters). The simultaneous solution of these equations was obtained in three Newton-Raphson iterations. The convergence tolerance was set to 0.001. The results are summarized in Table 3.

CONCLUSIONS

The inverse problem offers a rigorous mathematical technique for solving flow-constrained networks. The method is also explicit and computationally efficient. However, a serious problem in the development of such models is that their mathematical formulation may be ill-posed, which renders the solution unobtainable.

In this paper, necessary and sufficient conditions for the solvability of mathematical models of flow-constrained networks are presented. These conditions rely on the premise that a physically feasible solution exists. They are applicable to even-determined models of water-distribution systems. The mathematical well-posedness of the flow-constrained model is secured under these conditions.

It may appear that little flexibility exists in the selection of indeterminate edge parameters and corresponding flow specifications, i.e., only the non-JST pipes may be flow-constrained. That, however, is not exactly the case, because the user has some control over the solution process, i.e., the freedom to choose any set of pipes that constitutes a JST of the network. In most water-distribution networks, the number of possible JSTs is relatively large, due to the numerous loops present for reliability and water-quality purposes. As a result, a judiciously chosen JST may usually be found that would allow the targeted pipes to be flow-constrained.

ACKNOWLEDGMENTS

This material is based upon work partially supported by the National Science Foundation under grant ECE-861205. The support received is gratefully acknowledged.

APPENDIX I. NONSINGULARITY PROOF

Let $\mathbf{\Lambda} = [\mathbf{\Lambda}_1 \ \mathbf{\Lambda}_2]$ and $\mathbf{\Gamma} = [\mathbf{\Gamma}_1 \ \mathbf{\Gamma}_2]$, where $\mathbf{\Lambda}_1$ and $\mathbf{\Gamma}_1$, are n by n and $e - n$ by n matrices and $\mathbf{\Lambda}_2$ and $\mathbf{\Gamma}_2$ are n by $e - n$ and $e - n$ by $e - n$ matrices, respectively. Let us put the matrix $\mathbf{M}$ into the following form:

$$\mathbf{M} = \begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 \\ \mathbf{\Gamma}_1 & \mathbf{\Gamma}_2 \end{bmatrix} \tag{16}$$

where the n columns of $\mathbf{\Lambda}_1$ and $\mathbf{\Gamma}_1$ correspond to the edges of some spanning tree over the junction nodes of G and the datum-to-first-junction-node edge, that is, the JST; the $e - n$ columns of $\mathbf{\Lambda}_2$ and $\mathbf{\Gamma}_2$ correspond to the remaining $s - 1$ edges, i.e., the connections between junction and fixed-grade nodes, and the remaining non-JST edges.

Observation 1: Because $\mathbf{\Lambda}_1$ represents a JST, the first n rows (junction nodes) and columns (edges) of $\mathbf{M}$ may always be permuted so that $\mathbf{\Lambda}_1$ is a triangular matrix with values of ± 1 on its diagonal.

Observation 2: The remaining $e - n$ rows and columns of $\mathbf{M}$ may always be permuted so that $\mathbf{\Gamma}_2$ is a diagonal matrix with values of 1 on its diagonal.

For proofs of the foregoing, the reader is referred to Boulos and Altman (1993).

Let $\mathbf{M}^p$ denote the algebraic topological composition of the expanded Jacobian matrix of (11), constructed to solve up to $p \leq e - n$ additional unknowns (i.e., flow constraints). Note that for each additional unknown associated with an edge j, a column of zeroes will be appended to both $\mathbf{\Lambda}$ and $\mathbf{\Phi}$, while the additional column of $\mathbf{\Gamma}$ will be identical to its respective jth column. Then, $\mathbf{M}^p$ may be put in the following form:

$$\mathbf{M}^p = \begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 & \mathbf{\Lambda}_3^p \\ \mathbf{\Gamma}_1 & \mathbf{\Gamma}_2 & \mathbf{\Gamma}_3^p \\ \mathbf{\Phi}_1^p & \mathbf{\Phi}_2^p & \mathbf{\Phi}_3^p \end{bmatrix} \tag{17}$$

where $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ represent the continuity equations over the JST edges and the remaining edges, respectively; $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ represent the pseudo and fundamental circuits over the same edges. Note that by construction, $\mathbf{\Phi}_1^p$, $\mathbf{\Lambda}_3^p$, and $\mathbf{\Phi}_3^p$ = zero

J. Hydraul. Eng. 1995.121:427-431.

matrices of dimensions p by n, n by p, and p by p, respectively. Also, $\mathbf{\Lambda}_1$ = a lower (upper) triangular matrix with values of ± 1 on its diagonal. Observe that if $p = e - n$, then $\mathbf{\Gamma}_3^g = \mathbf{\Gamma}_2$ (modulo some permutation of the last $e - n$ columns of $\mathbf{M}^{(e-n)}$.

Theorem 1: The matrix $\mathbf{M}^{(e-n)}$ is nonsingular.

Proof: Through an appropriate permutation of the last $e - n$ rows of $\mathbf{M}^{(e-n)}$, we can force $\mathbf{\Phi}_2^g$ to be an $(e - n)$ by $(e - n)$ identity matrix. Because $\mathbf{\Phi}_1^g$ and $\mathbf{\Phi}_3^g$ are both zero matrices, we can zero out $\mathbf{\Lambda}_2$ and $\mathbf{\Lambda}_2$ without affecting $\mathbf{\Lambda}_1$, $\mathbf{\Lambda}_3^g$ and $\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_3^g$, and then the resulting matrix will have the following form:

$$\begin{bmatrix} \mathbf{\Lambda}_1 & 0 & 0 \\ \mathbf{\Gamma}_1 & 0 & \mathbf{I} \\ 0 & \mathbf{I} & 0 \end{bmatrix} \tag{18}$$

where $\mathbf{I}$ = identity matrix of appropriate dimension. Interchanging the last $(e - n)$ columns of $\mathbf{M}^{(e-n)}$ with those corresponding to the non-JST and fixed-grade edges results in a triangular matrix with values of ± 1 on its diagonal. Because the determinant of any triangular matrix is the product of its diagonal elements, it follows that the resulting matrix is nonsingular. This concludes the proof.

Corollary 1: The matrix $\mathbf{M}^p$, $1 \leq p \leq e - n$, is nonsingular.

It follows that the Jacobian of (11) is nonsingular and, therefore, that the inverse problem is solvable.

APPENDIX II. REFERENCES

Bhave, P. R. (1986). "Unknown pipe characteristics in Hardy-Cross method of network analysis." *J. Indian Water Works Assoc.*, India, 18(2), 133–135.

Bhave, P. R. (1988). "Calibrating water distribution network models." *J. Envir. Engrg.*, ASCE, 114(1), 120–136.

Bhave, P. R. (1990). "Rules for solvability of pipe networks." *J. Indian Water Works Assoc.*, 22(1), 7–10.

Bhave, P. R. (1991). *Analysis of flow in water distribution networks.* Technomic Publishing Co., Inc., Lancaster, Pa.

Boulos, P. F., and Altman, T. (1991). "A graph-theoretic approach to explicit nonlinear pipe network optimization." *J. Appl. Mathematical Modelling*, Vol. 15, 459–466.

Boulos, P. F., and Altman, T. (1993). "An explicit approach for modelling closed pipes in water networks." *J. Appl. Mathematical Modelling*, Vol. 17, 437–443.

Boulos, P. F., Altman, T., and Liou, J. C. P. (1993). "On the solvability of water distribution networks with unknown pipe characteristics." *J. Appl. Mathematical Modelling*, Vol. 17, 380–387.

Boulos, P. F., and Ormsbee, L. E. (1991a). "Explicit network calibration for multiple loading conditions." *J. Civ. Engrg. Systems*, Vol. 8, 153–160.

Boulos, P. F., and Ormsbee, L. E. (1991b). "A comprehensive algorithm for network calibration." *18th Annu. ASCE Water Resour. Conf.*, ASCE, New York, N.Y., 949–953.

Boulos, P. F., and Wood, D. J. (1990a). "Explicit calculation of pipe network parameters." *J. Hydr. Engrg.*, ASCE, 116(11), 1329–1344.

Boulos, P. F., and Wood, D. J. (1990b). *KYPIPED: A comprehensive pipe distribution network analyzer.* Civ. Engrg. Software Ctr., Univ. of Kentucky, Lexington, Ky.

Boulos, P. F., and Wood, D. J. (1991a). "An explicit algorithm for calculating operating parameters for water networks." *J. Civ. Engrg. Systems*, Vol. 8, 115–122.

Boulos, P. F., and Wood, D. J. (1991b). "Explicit determination of network parameters for improving water distribution system performance." *2nd Int. Conf. on Computer Methods and Water Resour.*, Rabat, Morocco.

Boulos, P. F., and Wood, D. J. (1992). "Closure to 'Explicit calculation of pipe network parameters.'" *J. Hydr. Engrg.*, ASCE, 118(7), 1062–1063.

Gofman, E., and Rodeh, M. (1981). "Loop equations with unknown pipe characteristics." *J. Hydr. Div.*, ASCE, 107(9), 1047–1060.

Jowitt, P. W., and Xu, C. (1990). "Optimal valve control in water distribution networks." *J. Water Resour. Plng. and Mgmt.*, ASCE, 116(4), 455–472.

Liggett, J. A. (1993). "Network monitoring and the algorithmic location of leaks under steady and unsteady conditions." *Water supply systems: State-of-the-art and future trends*, E. Cabrera and F. Martinez, eds., Computational Mechanics Publ., Bell & Bain Ltd., Glasgow, U.K.

Ormsbee, L. E., and Wood, D. J. (1986a). "Hydraulic design algorithms for pipe networks." *J. Hydr. Engrg.*, ASCE, 112(12), 1195–1207.

Ormsbee, L. E., and Wood, D. J. (1986b). "Explicit pipe network calibration." *J. Water Resour. Plng. and Mgmt.*, ASCE, 112(2), 166–182.

Osiadacz, A. J. (1987). *Simulation and Analysis of Gas Networks.* E&F Spoon Ltd., London, England.

Pudar, R. S., and Liggett, J. A. (1992). "Leaks in pipe networks." *J. Hydr. Engrg.*, ASCE, 118(7), 1031–1046.

Shamir, U., and Howard, C. D. D. (1968). "Water distribution system analysis." *J. Hydr. Div.*, ASCE, 94(1), 219–234.

Shamir, U., and Howard, C. D. D. (1970). "Closure to 'Water distribution system analysis.'" *J. Hydr. Div.*, ASCE, 96(2), 577–578.

Shamir, U., and Howard, C. D. D. (1977). "Engineering analysis of water distribution systems." *J. Am. Water Works Assoc.*, 69(9), 510–514.

Sterling, M., and Bargiela, A. (1984). "Leakage reduction by optimised control of valves in water networks." *Trans., Inst. of Measurement and Control*, London, England, 6(6), 293–298.

APPENDIX III. NOTATION

The following symbols are used in this paper:

c = number of flow constraints;
e = number of edges;
F = function;
$\mathbf{H}$ = piezometric head (L);
l = number of fundamental loops;
$\mathbf{M}$ = topological coefficient matrix;
n = number of junction nodes;
p = number of indeterminate parameters;
$\mathbf{Q}$ = volumetric flow rate (L^3T^{-1});
$\mathbf{q}$ = external nodal demand (L^3T^{-1});
q_c = set flow (L^3T^{-1});
s = number of fixed-grade nodes;
$\mathbf{\Gamma}$ = edge-loop incidence matrix;
σ = exponent of flow rate;
$\mathbf{\Phi}$ = Boolean matrix;
ϕ = function;
$\mathbf{\Psi}$ = headloss (L); and
$\mathbf{\Lambda}$ = edge-junction node incidence matrix.

J. Hydraul. Eng. 1995.121:427-431.