Convergence of Newton Method in Nonlinear Network Analysis

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Abstract—The convergence of Newton method in nonlinear network analysis is discussed. It is shown that formulating the nonlinear edge flow resistances in terms of edge flow rates (Flow method) or corrected loop flow rates (Loop method) gives better convergence than the nodal head or pressure formulation (Node method). This evaluation is based on the estimation of the magnitude of the radius of attraction for the Newton method. Our result applies to both water and low pressure gas networks.

Keywords—Fluid flow, Nonlinear hydraulic networks, Newton method.

INTRODUCTION

The ability to evaluate the performance of hydraulic networks is necessary for proper design, to understand the effects of dynamic changes, and to ensure meeting the demand requirements within acceptable operational limits. Over the past years, the analysis of network flow and pressure distribution has been an active, quite broadly based area of research and interest, and various network analysis algorithms have been reported [1-4]. These have been expressed in terms of the unknown edge flow rates (Flow method), corrected loop flow rates (Loop method), and nodal heads or pressures (Nodal method). Owing to the presence of nonlinearity, neither of the network analysis formulations can be solved directly; hence, an iterative procedure must be employed. Several iterative techniques for the solution of the flow, loop, and nodal equations have been suggested and, from which, the Newton method is the most widely used.

A comprehensive comparison of the reliability and performance of the Newton-based algorithms has been previously reported by Wood and Rayes [2]. An extensive data base was utilized in their study. The majority of the data was provided by consulting engineers and water utilities and represents actual or proposed (as opposed to sample or contrived) water distribution systems from all over the United States. They documented several reliability problems associated with the nodal formulation, while both the flow and loop formulations proved to be stable and consistently provided reliable results with successful convergence being achieved in all cases tested. They noted that the nodal formulation exhibits a relatively slow convergence rate, is highly sensitive to the starting values, and is unable to handle low resistance lines (mainly short lengths of large

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diameter mains). Shamir and Howard [5] also noted the possibility of convergence difficulties associated with the nodal formulation such as oscillation or divergence. Others reported similar observations; mainly that the nodal formulation suffers from poor convergence characteristics with extreme sensitivity with respect to the starting values [1,4,6,7]. Both Wood and Rayes [2] and Nielsen [6] discouraged the use of the nodal formulation in network analysis.

In this paper, the convergence of Newton-based methods for nonlinear network analysis is investigated. This convergence problem has not been addressed to date. It is shown that the flow and loop formulations exhibit superior convergence characteristics over the nodal formulation. This evaluation is based on the estimation of the magnitude of the radius of attraction for the Newton method. Our result applies to both water and low pressure gas networks.

NETWORK TOPOLOGY

A distribution network may be represented by a directed connected graph \( G = (N \cup S, E) \) with \( n = |N| \) junction nodes, \( s = |S| \) fixed grade nodes, and \( e = |E| \) edges. Here, \( |U| \) denotes the cardinality of the set \( U \). \( G \) need not be planer, i.e., edges may cross in plan or elevation and yet not join. A graph is directed if all of its edges are assigned an orientation, and it is connected if it contains a path between any pair of nodes. A path is a sequence of edges that allows the movement of flow from one node to another. A junction node is a terminus of one or more edges with an associated demand where flow can enter or exit the graph. A fixed grade node is a point of known head or pressure (e.g., well, treatment plant, lake, reservoir or storage facility, constant pressure region). The graph is rooted at any fixed grade node, which is then referred to as the datum node. The edge-node interconnection uniquely defines the network topology with all independent loops being identified. Each loop is either a fundamental or a pseudo-loop. A fundamental loop is a closed path of connected edges that uniquely contains exactly one edge that no other loop possesses. A pseudo-circuit is a simple path of connected edges between the datum node and any other fixed grade node. A simple path is a path in which no node is traversed more than once. The algebraic equilibrium relation describing the network topology is given by

\[
e = n + m,
\]

in which \( m \) is the number of loops (fundamental as well as pseudo). For a detailed discussion of the above, the reader is referred to Osiadacz [4] and Boulos and Altman [8].

MECHANICS OF NETWORK FLOW

The problem of stationary flow distribution in pipe networks is characterized by two physical laws. Applying Kirchhoff's first law, the nodal equations are

\[
q_v + \sum_{u=1}^{e} \lambda_{uv} Q_u = 0; \quad v = 1, \ldots, n \quad \text{and} \quad \lambda_{uv} \in \{-1, 0, 1\},
\]

implying that at each junction node \( v \), the sum of inflows (\( \lambda_{uv} = -1 \)) and outflows (\( \lambda_{uv} = 1 \)) is zero. Here, \( Q_u \) is the volumetric flow rate in edge \( u \); and \( q_v \) is the external load at junction node \( v \) (if inflow). Applying Kirchhoff's second law, the loop equations are

\[
\Phi_w + \sum_{u=1}^{e} \gamma_{wu} \Delta \Psi_u = 0; \quad w = 1, \ldots, m \quad \text{and} \quad \gamma_{wu} \in \{-1, 0, 1\},
\]

asserting that around any loop \( w \), the algebraic sum of head (pressure if the fluid is gas) drops is equal to the head (pressure) differences between the two boundary nodes \( \Phi_w \) (zero for fundamental loops). The head (pressure) drop or flow resistance term \( \Delta \Psi \) is considered positive (\( \gamma_{wu} = 1 \))
when the flow is oriented in the same direction as the positive edge orientation. For each edge, the flow resistance is a nonlinear characteristic function relating the flow rate to the head (pressure) drop across it. We consider the quadratic functional relation of the form

$$\Delta \Psi_u = \xi_u Q^2_u = \Psi_{ui} - \Psi_{uj},$$

(4)

and its dual formulation

$$Q_u = \left( \frac{\Psi_{ui} - \Psi_{uj}}{\xi_u} \right)^{1/2},$$

(5)

in which $\xi_u$ is the resistance of edge $u$ that is a function of the edge length, diameter, and material, and on the fluid (water or gas); and $\Psi_{ui}$ and $\Psi_{uj}$ are the heads (pressures) at the ends of edge $u$.

### BASIC NETWORK ALGORITHMS

#### Analytical Formulation

The analytical network flow problem may be formulated in terms of edge flow rates, loop flow correction factors, or nodal heads (pressures).

**FLOW METHOD.** Equation (4) is used to write equation (3) in terms of edge flow rates, which is then combined with equation (2) to give a set of $e$ quasi-linear equations with $e$ unknown edge flow rates as follows $[1-3, 8-13]$,

$$q_u + \sum_{u=1}^{e} \lambda_{uu} Q_u = 0; \quad v = 1, \ldots, n,$$

$$\Phi_w + \sum_{u=1}^{e} \gamma_{uu} \xi_u Q^2_u = 0; \quad w = 1, \ldots, m.$$  

(6)

The flow analysis method solves equation (6) for the edge flow rates. The nodal heads (pressures) are then determined directly from equation (4).

**LOOP METHOD.** The method solves for corrective flow rates around the loops of the network without violating nodal continuity $[1-4, 14]$. A loop flow correction factor $\delta Q$, is applied to every edge in its corresponding loop to give a corrected edge flow rate $Q'$ as follows:

$$Q'_u = Q_u + \sum_{u=1}^{m} \gamma_{uu} \xi_u Q^2_u; \quad u = 1, \ldots, e.$$  

(7)

Substituting for $Q'$ from equation (7), equation (4) becomes

$$\Delta \Psi_u = \xi_u Q'^2_u.$$  

(8)

Substituting equation (8) in (3) gives

$$\Phi_w + \sum_{u=1}^{e} \gamma_{uu} \xi_u Q'^2_u = 0; \quad w = 1, \ldots, m,$$

(9)

which is a system of $m$ nonlinear equations with $m$ unknown loop flow correction factors. The loop analysis method solves equation (9) for the loop flow correction factors and, hence, the corrected edge flow rates. The head (pressure) at each junction node can then be derived from equation (4) with known edge flow rates.

**NODAL METHOD.** Equation (5) is used to write equation (2) in terms of nodal heads (pressures) as follows $[1-5]$:

$$q_v + \sum_{u=1}^{e} \lambda_{uu} \left( \frac{\Psi_{ui} - \Psi_{uj}}{\xi_u} \right)^{1/2} = 0; \quad v = 1, \ldots, n,$$

(10)

resulting in a system of $n$ nonlinear equations with $n$ unknown heads (pressures). The nodal analysis method solves equation (10) to give the nodal heads (pressures). The flow rate in each edge can then be obtained through a direct application of equation (5) with known heads (pressures) at the edge endpoints.
NUMERICAL SOLUTION

Owing to the presence of nonlinearity, neither of the network analysis formulations can be solved directly, and an iterative procedure must be employed. Iterative solutions for edge flow rate, corrected edge flow rate, or nodal head may be obtained using Newton's method. Convergence of this method is always guaranteed whenever the starting solution lies within a region $R$, called the "region of attraction," i.e., the method is locally convergent. In order to adequately evaluate the convergence characteristics of the three network analysis formulations, a bound on the radius of attraction for each formulation should be estimated.

NEWTON-FLOW METHOD. This method is also known in the U.S. water works industry as the "Linear" or "Flow adjustment" method. Without loss of generality [6], assume $n < e \leq 2n$. Let $x_i$, $i = 1, \ldots, e$ denote the $e$ unknown edge flow rates; let $p$'s, $q$'s, and $\xi$'s be fixed constants.

We have the following $n$ linear and $m$ nonlinear equations, shown in equations (11) and (12), respectively.

\[
\begin{align*}
\lambda_{11} x_1 + \lambda_{12} x_2 + \cdots + \lambda_{1e} x_e + q_1 &= 0, \\
\lambda_{21} x_1 + \lambda_{22} x_2 + \cdots + \lambda_{2e} x_e + q_2 &= 0, \\
&\vdots \\
\lambda_{n1} x_1 + \lambda_{n2} x_2 + \cdots + \lambda_{ne} x_e + q_n &= 0.
\end{align*}
\]

\[
\begin{align*}
\gamma_{11} x_1^2 + \gamma_{12} x_2^2 + \cdots + \gamma_{1e} x_e^2 + \Phi_1 &= 0, \\
\gamma_{21} x_1^2 + \gamma_{22} x_2^2 + \cdots + \gamma_{2e} x_e^2 + \Phi_2 &= 0, \\
&\vdots \\
\gamma_{m1} x_1^2 + \gamma_{m2} x_2^2 + \cdots + \gamma_{me} x_e^2 + \Phi_m &= 0.
\end{align*}
\]

Combining equations (11) and (12), we have a system of $e$ equations with $e$ unknowns. Solving equations (11) for $x_1, x_2, \ldots, x_t$, $t \leq n$, we obtain

\[
\begin{align*}
x_1 &= \eta_{11} x_{t+1} + \eta_{12} x_{t+2} + \cdots + \eta_{1e} x_e + a_1, \\
x_2 &= \eta_{21} x_{t+1} + \eta_{22} x_{t+2} + \cdots + \eta_{2e} x_e + a_2, \\
&\vdots \\
x_t &= \eta_{1t} x_{t+1} + \eta_{2t} x_{t+2} + \cdots + \eta_{et} x_e + a_t,
\end{align*}
\]

where $t \leq e - n$.

Substituting equations (13) in (12) and changing the notation of the variables, we have

\[
\begin{align*}
f_1 &= p_{11} x_1^2 + p_{12} x_2^2 + \cdots + p_{1l} x_l^2 + b_{11} x_1 + b_{12} x_2 + \cdots + b_{1l} x_l + d_1 = 0, \\
f_2 &= p_{21} x_1^2 + p_{22} x_2^2 + \cdots + p_{2l} x_l^2 + b_{21} x_1 + b_{22} x_2 + \cdots + b_{2l} x_l + d_2 = 0, \\
&\vdots \\
f_l &= p_{l1} x_1^2 + p_{l2} x_2^2 + \cdots + p_{ll} x_l^2 + b_{l1} x_1 + b_{l2} x_2 + \cdots + b_{ll} x_l + d_l = 0,
\end{align*}
\]

where $x_1, x_2, \ldots, x_l$ stand lieu of $x_{t+1}, x_{t+2}, \ldots, x_e$ and $l = e - t$.

System (14) can now be written as

\[
P(\bar{x}) = 0,
\]

where $P = (f_1, f_2, \ldots, f_l)^T$, $\bar{x} = [x_1, x_2, \ldots, x_l]^T$.

Denote by

\[
P'(\bar{x}) = \frac{d(f_1(\bar{x}), \ldots, f_l(\bar{x}))}{d(x_1, \ldots, x_l)} = \left[ \frac{\partial f_i}{\partial x_j} \right], \quad i, j = 1, \ldots, l,
\]

the Jacobian of $P$ at $\bar{x}$, where $\frac{\partial f_i}{\partial x_j} = 2p_{ij} x_j + b_{ij}$ for $i, j = 1, \ldots, l$. 

The second derivative is

$$P''(\vec{x}) = \left[ \frac{\partial^2 f_k}{\partial x_i \partial x_j} \right], \quad i, j, k = 1, \ldots, l,$$

which can be considered as a bilinear form

$$P''(\vec{x}) \vec{x}' \vec{x}'' = \sum_{i,j=1}^{l} \frac{\partial^2 f_k(\vec{x})}{\partial x_i \partial x_j} x_i' x_j'', \quad k = 1, \ldots, l.$$

Hence, we get the following estimate on its norm

$$||P''(\vec{x})|| \leq \max_{k,l,j} \sum_{i,j=1}^{l} \left| \frac{\partial^2 f_k(\vec{x})}{\partial x_i \partial x_j} \right| x_i' x_j'' \leq 2l^2 \max_{i,j} |p_{ij}| = 2l^2 L, \quad (16)$$

where

$$L = \max_{i,j} |p_{ij}|.$$

Let us put

$$||\vec{x}|| - \max_{j} |x_j|, \quad \text{where} \quad \vec{x} = [x_1, \ldots, x_l]^T.$$

Let \( \vec{x}_0 = [x_1^0, \ldots, x_l^0]^T \) be an initial approximate solution to System (14). Then the Newton method can be written as follows

$$\vec{x}_1 = \vec{x}_0 - [P''(\vec{x}_0)]^{-1} P(\vec{x}_0), \quad (17)$$

$$\vec{x}_{i+1} = \vec{x}_i - [P''(\vec{x}_i)]^{-1} P(\vec{x}_i), \quad (18)$$

Here, \( i \) designates the Newton iteration number. Equations (17) and (18) can be rewritten as

$$P'(\vec{x}_0)(\vec{x}_1 - \vec{x}_0) = -P(\vec{x}_0), \quad (19)$$

$$P'(\vec{x}_i)(\vec{x}_{i+1} - \vec{x}_i) = -P(\vec{x}_i), \quad (20)$$

respectively. Hence, using equations (14) and (15), we obtain the following

$$\frac{\partial f_1}{\partial x_1}(\vec{x}_i)(x_1^{i+1} - x_1) + \frac{\partial f_1}{\partial x_2}(\vec{x}_i)(x_2^{i+1} - x_1) + \cdots + \frac{\partial f_1}{\partial x_1}(\vec{x}_i)(x_1^{i+1} - x_1) - f_1 = 0, \quad \text{for} \ i = 1, \ldots, l,$$

$$\frac{\partial f_2}{\partial x_1}(\vec{x}_i)(x_1^{i+1} - x_1) + \frac{\partial f_2}{\partial x_2}(\vec{x}_i)(x_2^{i+1} - x_2) + \cdots + \frac{\partial f_2}{\partial x_1}(\vec{x}_i)(x_1^{i+1} - x_1) - f_2 = 0, \quad (21)$$

$$\vdots$$

$$\frac{\partial f_l}{\partial x_1}(\vec{x}_i)(x_1^{i+1} - x_1) + \frac{\partial f_l}{\partial x_2}(\vec{x}_i)(x_2^{i+1} - x_2) + \cdots + \frac{\partial f_l}{\partial x_1}(\vec{x}_i)(x_1^{i+1} - x_1) - f_l = 0,$$

where \( \vec{x}_0 = [x_1^0, \ldots, x_l^0]^T \) and \( \vec{x}_i = [x_1, \ldots, x_l]^T. \)

**Theorem 1.** Let the functions \( f_1, f_2, \ldots, f_l \) and the initial approximation \( \vec{x}_0 \) satisfy the following conditions:

1. \( |f_j(x_1^0, \ldots, x_l^0)| \leq \eta \quad (j = 1, \ldots, l), \)
2. The Jacobian

$$\left[ \frac{\partial f_j}{\partial x_k}(x_1^0, x_2^0, \ldots, x_l^0) \right], \quad j, k = 1, \ldots, l,$$
has a determinant $D \neq 0$ and if $A_{i,k}$ is the cofactor of the element $\frac{\partial f_i}{\partial x_k}$ at $\vec{x}_0$, then the following inequality holds,

\begin{equation}
\frac{1}{|D|} \sum_{j=1}^{l} |A_{j,k}| \leq B, \quad k = 1, \ldots, l,
\end{equation}

or, by equation (16), $2l^2 \max_{i,j} |p_{ij}| \leq 2l^2 L$,

\begin{equation}
h = B^2 \eta \leq \frac{1}{2}.
\end{equation}

Then System (14) has a solution

$$\vec{z}^* = (x_1^*, x_2^*, \ldots, x_l^*)^T,$$

lying in the sphere with center $\vec{x}_0$ and a radius $r$, where

$$r \geq \frac{1 - \sqrt{1 - 2h}}{h} B \eta,$$

and the Newton method given by equations (18) or (20) converges to $\vec{z}^*$.

PROOF. The proof follows from the proof of Theorem 1 in [15, (3.XVIII)].

NEWTON-LOOP METHOD. This method is also known in the U.S. water works industry as the “Simultaneous Path Adjustment” or “Newton-Raphson Loop” method. From equations (7) and (9), it follows that the upper bound on the norm of the Hessian given by inequality (16) also applies to the Loop method. In fact, Nielsen [6] showed that the successive values for edge flow rates obtained by iterating on the flow system of equations (6) are identical to the values obtained by iterating on the loop system of equation (9).

NEWTON-NODAL METHOD. This method is also known in the U.S. water works industry as the “Simultaneous Node Adjustment” or “Newton-Raphson Node” method. System (10) can be written as

$$g_1 = q_1 + \sum_{u=1}^{e} \lambda_{1u} \left( \frac{\Psi_{u1} \Psi_{u2}}{\xi_u} \right)^{1/2} = 0,$$

$$g_2 = q_2 + \sum_{u=1}^{e} \lambda_{2u} \left( \frac{\Psi_{u1} \Psi_{u2}}{\xi_u} \right)^{1/2} = 0,$$

$$\vdots$$

$$g_m = q_m + \sum_{u=1}^{e} \lambda_{mu} \left( \frac{\Psi_{u1} \Psi_{u2}}{\xi_u} \right)^{1/2} = 0,$$

or in a more compact form

$$G(\vec{\Psi}) = \vec{0},$$

where $G = (g_1, g_2, \ldots, g_m)^T$ and $\vec{\Psi} = [\Psi_1, \Psi_2, \ldots, \Psi_m]^T$.

Alternatively, from equations (2) and (5), equation (23) may be written as

$$G(\vec{\Psi}(\vec{\Psi})) = \vec{0},$$
with
\[
\frac{\partial Q_u}{\partial \Psi_{ui}} = \frac{1}{2\sqrt{\xi_u(\Psi_{ui} - \Psi_{uj})}},
\]
(25)
and
\[
\frac{\partial^2 Q_u}{\partial \Psi_{ui} \partial \Psi_{uj}} = -\frac{1}{4\sqrt{\xi_u}} (\Psi_{ui} - \Psi_{uj})^{-3/2}.
\]
(26)

It is clear that for the flow and loop formulations, given by equations (6) and (9), respectively, the norm of the matrix of second derivatives is always bounded from above. That, of course, is not the case with the nodal formulation, given by equation (10). There, the norm of the matrix of second derivatives depends on \(\overline{\Psi}_0\) (as can be seen from equation (26)) and may become too large to satisfy Condition (4) of Theorem 1, unless \(\overline{\Psi}_0\) is very close to the solution or the \(\xi\) values are relatively large.

In practice, it is not always feasible to explicitly check for the satisfiability of the sufficiency conditions of Theorem 1. It is clear, however, that the flow and loop formulations possess better lower bounds on the radius of attraction, are less sensitive to the starting values and the edge resistances and, hence, exhibit better convergence characteristics than the nodal formulation. This is in agreement with the previously cited studies.

CONCLUSION

Algorithms for hydraulic network analysis based on different system-variable formulations were proposed by a number of researchers. Convergence problems were frequently observed with the nodal formulation. Two of the most important convergence difficulties encountered were attributed to edges with relatively low resistances and to improper selection of initial nodal head values. In this paper, the convergence conditions of the three network analysis formulations were examined. The study indicates that the flow and loop formulations can be effectively used to overcome at least some of the convergence problems associated with the nodal formulation.

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