Computational Complexity CSC 5802

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Final Report

Fall

Introduction:

The **** is one of the most challenging combinatorial optimization tasks. Defined more than 40 years ago, this problem consists in designing the optimal set of routes for fleet of vehicles in order to serve a given set of customers. The interest in VRP is motivated by its practical relevance as well as by its considerable difficulty.

The VRPwas arisen naturally byDantzingand Ramserin 1959 as a central problem in the fields of transportation, distribution and logistics. In some market sectors, transportation means a high percentage of the value added to goods. Therefore, the utilization of computerized methods for transportation often results in significant savings ranging from 5% to 20% in the total costs, as reported in [Toth& Vigo 2001]. Usually, in real world VRPs, many side constraints appear. Some of the most important restrictions are:

The Vehicle Routing Problem is a generic name given to a whole class of problems in which a set of routes for a fleet of vehicles based at one or several depots must be determined for a number of geographically dispersed cities or customers. The objective of the VRP is to deliver a set of customers with known demands on minimum-cost vehicle routes originating and terminating at a depot. In the two figures below we can see a picture of a typical input for a VRP problem and one of its possible outputs:

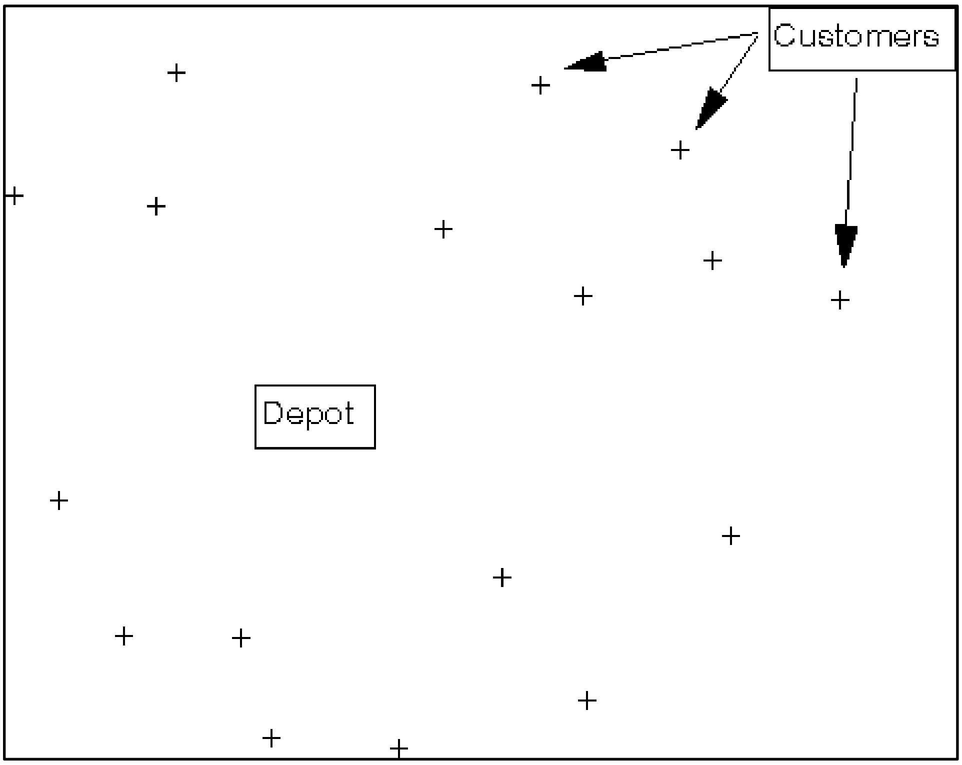


Figure 1. Typical input for a Vehicle Routing Problem

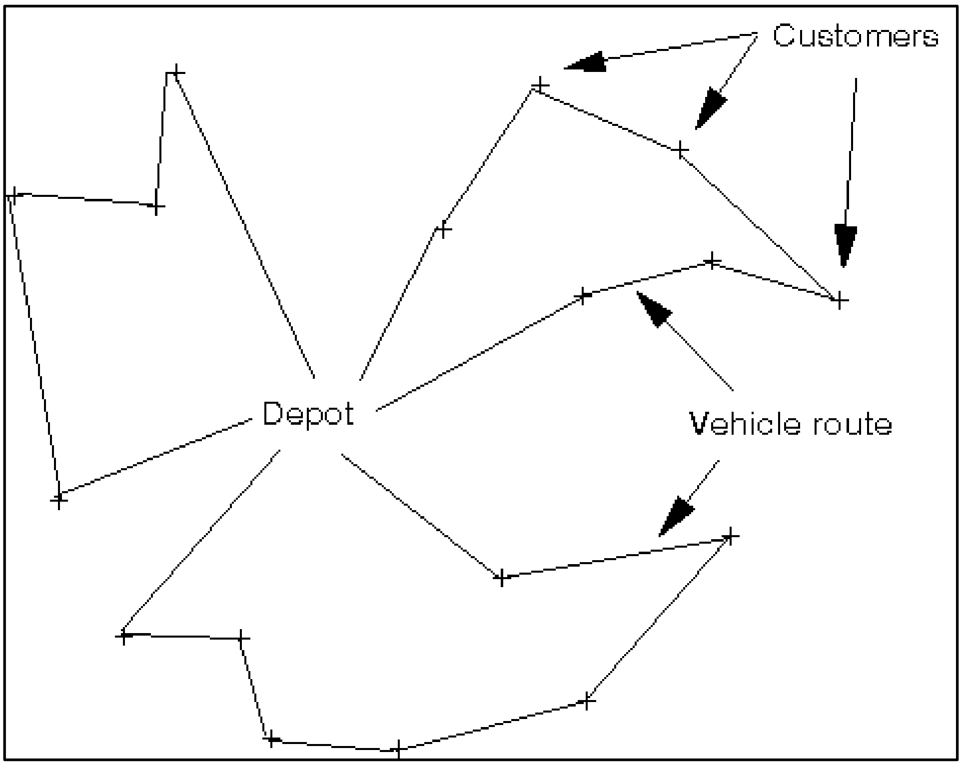


Figure 2. An output for the instance above

Definition:

The VRP is a combinatorial problem whose ground set is the edges of a graph ***G(V,E)***. The notation used for this problem is as follows:

- $V = \{v_0, v_1, \ldots, v_n\}$ is a vertex set, where:
    - Consider a depot to be located at v_0 .
    - Let $V' = V \setminus \{v_0\}$ be used as the set of n cities.
- $A = \{(v_i, v_j) / v_i, v_j \in V; I \neq j$ is an arc set.
- C is a matrix of non-negative *costs* or *distances* c_{ij} between customers v_i and v_j.
- d is a vector of the customer demands.
- R_i is the route for vehicle i.
- m is the number or vehicles (all identical). One route is assigned to each vehicle.

When $c_{ij} = c_{ji}$ for all $(v_i, v_j) \in A$ the problem is said to be symmetric and it is then common to replace A with the edge se $E = \{(v_i, v_j) | v_i, v_j \in V; i < j\}$

With each vertex v_i in V' is associated a quantity q_i of some goods to be delivered by a vehicle. The VRP thus consists of determining a set of m vehicle routes of minimal total cost, starting and ending at a depot, such that every vertex in V' is visited exactly once by one vehicle.

For easy computation, it can be defined $b(V) = \lceil(\sum_{v_i \in V} d_i)/C\rceil$, an obvious lower bound on the number of trucks needed to service the customers in set ***V***.

We will consider a service time(δ_i time needed to unload all goods), required by a vehicle to unload the quantity at q_i It is v_i required that the total duration of any vehicle route (travel plus service times) may not surpass a given bound , so, in this D context the cost is taken to be c_{ij} the travel time between the cities. The VRP defined above is NP-hardaccording to Lenstra andRinnooyKan 1981.

A feasible solution is composed of:

- a partition of V; $R_1, \ldots, R_m$
- a permutation of σ_i specifying $R_i \bigcup 0$ the order of the customers on route . i

The cost of a given route ($R_i = \{v_0, v_1, \ldots, v_{m+1}\}$) , where $v_i \in V$ and (**0** $v_0 = v_{m+1} = 0$ denotes the depot), is given by:

$$C(R_i) = \sum_{i=0}^{m} c_{i,i+1} + \sum_{i=1}^{m} \delta_i \quad .$$

A route is R_i feasible if the vehicle stop exactly once in each customer and the total duration of the route does not exceed a prespecifiedbound : . D $C(R_i) \leq D$

Finally, the cost of the problem solution ***S*** is: $F_{VRP}(S) = \sum_{i=1}^{m} C(R_i)$.

Some of VRP's Applications in the Real World:

- Mail Delivery and Collection
- Distribution of Goods to the stores

- Collection of coins from mail boxes
- Waste Collection
- School Bus Routing

The Classification of the Problem:

The VRP is a well known integer programming problem which falls into the category of NP Hard problems, meaning that the computational effort required to solve this problem increases exponentially with the problem size. For such problems it is often desirable to obtain approximate solutions, so they can be found fast enough and are sufficiently accurate for the purpose. Usually this task is accomplished by using various heuristic methods, which rely on some insight into the problem nature.

This difficult combinatorial problem conceptually lies at the intersection of these two well-studied problems:

- The Traveling Salesman Problem (TSP): If the capacity of the vehicles C is infinite, we can get an instance of the Multiple Traveling Salesman Problem (MTSP). An MTSP instance can be transformed into an equivalent TSP instance by adjoining to the graph ***k-1*** (being ***k*** the number of routes) additional copies of node**0** and its incident edges (there are no edges among the ***k*** depot nodes).
- The Bin Packing Problem (BPP): The question of whether there exists a feasible solution for a given instance of the VRP is an instance of the BPP. The decision version of this problem is conceptually equivalent to a

VRPmodel in which all edge costs are taken to be zero (so that all feasible solutions have the same cost).

Hence, we can think of the first transformation as relaxing the underlying packing (BPP) structure and the second transformation as relaxing the underlying routing (TSP) structure. A feasible solution to the full problem is a TSP tour (in the expanded graph) that also satisfies the packing constraints (i.e., the total demand along each of the ***k*** segments joining successive copies of the depot does not exceed ***C***). Because of the interplay between the two underlying models (both of them are NP Hard problems), instances of the Vehicle Routing Problem can be extremely difficult to solve in practice.

Does VRP an NP-Complete Problem?

If the capacity of the vehicles C is infinite, we can get an instance of the Multiple Traveling Salesman Problem (MTSP) which means that VRP has a special case of TSP as a subproblem**TSP ≤p VRP**and since TSP is an NP-Complete problem then VRP is NP-Complete Problem Also.

The Forms of the VRP Problem:

Usually, in real world VRPs, many side constraints appear. These constraints form the variations of the VRP problem. The following are the most common of these variations with their objective, feasibility, and formulation:

Capacitated VRP (CPRV):

CVRP is a Vehicle Routing Problem (VRP) in which a fixed fleet of delivery vehicles of uniform capacity must service known customer demands for a single commodity from a common depot at minimum transit cost. That is, CVRP is like VRP with the additional constraint that every vehicle must have uniform capacity of a single commodity.

- **Objective:** The objective is to minimize the vehicle fleet and the sum of travel time, and the total demand of commodities for each route may not exceed the capacity of the vehicle which serves that route.

- **Feasibility:** A solution is feasible if the total quantity assigned to each route does not exceed the capacity of the vehicle which services the route.

- **Formulation**: Let denote Q the capacity of a vehicle. Mathematically, a solution for the CVRP is the same that VRP's one, but with the additional restriction that the total demand of all customers supplied on a route R_i does not exceed the vehicle capacity : Q $\sum_{i=1}^{m} d_i \le Q$

VRP with Time Windows (VRPTW):

The VRPTW is the same problem that VRP with the additional restriction that in VRPTW a time window is associated with each customer $v \in V$ defining an interval $[e_v, l_v]$ wherein the customer has to be supplied. The interval $[e_0, l_0]$ the depot is called the scheduling horizon.

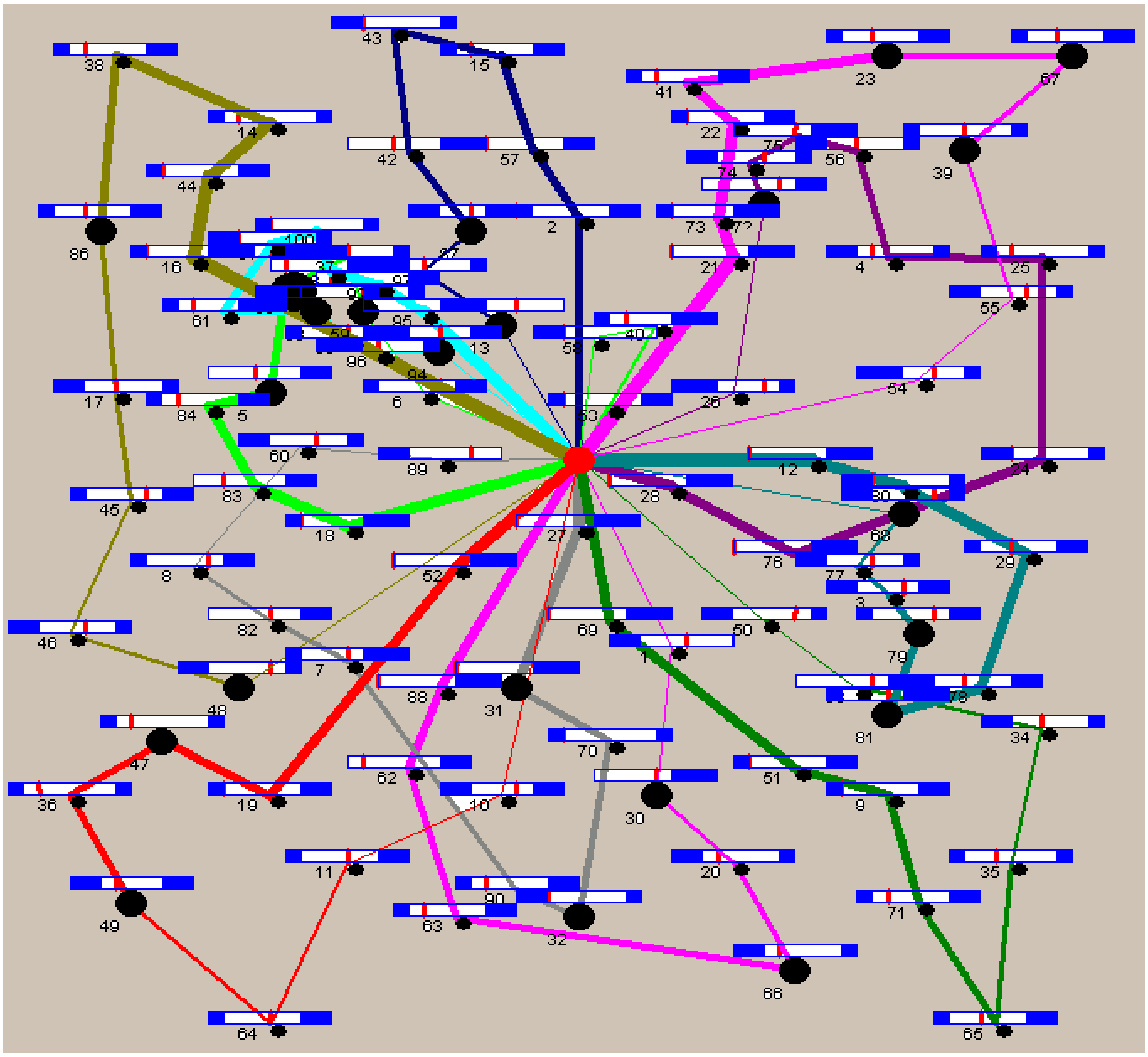
- **Objective:** The objective is to minimize the vehicle fleet and the sum of travel time and waiting time needed to supply all customers in their required hours.
- **Feasibility:** The VRPTW is, regarding to VRP, characterized by the following additional restrictions:
    - A solution becomes infeasible if a customer is supplied after the upper bound of its time window.

- A vehicle arriving before the lower limit of the time window causes additional waiting time on the route.
- Each route must start and end within the time window associated with the depot.
- In the case of soft time widows, a later service does not affect the feasibility of the solution, but is penalized by adding a value to the objective function.

- **Formulation:** Let b_i denote the beginning of service at customer i. Now for a route $R_i = (v_0, v_1, ..., v_m, v_{m+1})$ to be feasible it must additionally hold

  $e_{v_i} \le b_{v_i} \le l_{v_i}$, $1 \le i \le m$ and $b_{v_m} + \delta_{v_m} + c_{v_m,0} \le l_0$. Provided that a vehicle travels to the next customer as soon as it has finished service at the current customer, b_{v_i} can be recursively computed as $b_{v_i} = \max\{e_{v_i}, b_{v_{i-1}} + \delta_{v_{i-1}} + c_{v_{i-1},v_i}\}$ with $b_0 = e_0$ and $\delta_0 = 0$. Thus, a waiting time $w_{v_i} = \max\{0, b_{v_i} - b_{v_{i-1}} - \delta_{v_{i-1}} - c_{i-1,i}\}$ may be induced at customer v_i. The cost of route i is now given by $C_{VRPTW}(R_i) = \sum_{i=0}^{m} c_{i,i+1} + \sum_{i=1}^{m} \delta_i + \sum_{i=1}^{m} w_{v_i}$. For a solution S with routes $R_1, ..., R_{|S|}$, the cost of S is given by $C_{VRPTW}(S) = \sum_{i=1}(C_{VRPTW}(R_i) + M)$, where M is a large constant. M is added because minimization of the fleet size is considered to be the primary objective of the VRPTW. S is said to be feasible if all routes belonging to S are feasible and its customer is served by exactly one route. As described by Solomon [Solomon 1995], we assume that initially all vehicles leave the depot at the earliest possible time e_0. Having obtained a solution of the VRPTW, we adjust the depot departure time of each vehicle to eliminate any unnecessary waiting time.

In the figure below we can see a graph representing an instance for solving with VRPTW. There, blue and white bars are representing the time window (i.e. business hours), where white area represents when we can make a delivery at that customer. By other side, the red line shows when is the delivery made for this particular solution.



Multiple Depots VRP (MDVRP):

A company may have several depots from which it can serve its customers. If the customers are clustered around depots, then the distribution problem should be modeled as a set of independent VRPs. However, if the customers and the depots are intermingled then a Multi-Depot Vehicle Routing Problem should be solved.

A MDVRP requires the assignment of customers to depots. A fleet of vehicles is based at each depot. Each vehicle originates from one depot, service the customers assigned to that depot, and return to the same depot.

The objective of the problem is to service all customers while minimizing the number of vehicles and travel distance.

- **Objective:** The objective is to minimize the vehicle fleet and the sum of travel time, and the total demand of commodities must be served from several depots.

- **Feasibility:** A solution is feasible if each route satisfies the standard VRP constraints and begins and ends at the same depot.

- **Formulation:** The VRP problem is extended to the case wherein we have multiple depots, so we will note the vertex set like $V = \{v_1, \dots, v_n\} \cup V_0$, where $V_0 = \{v_{01}, \dots, v_{0d}\}$ are the vertex representing the depots. Now, a route i is defined by $R_i = \{d, v_1, \dots, v_m, d\}$, with $d \in V_0$. The cost of a route is calculated like in the case of the standard VRP.

VRP with Pick-Up and Delivering:

The Vehicle Routing Problem with Pick-up and Delivering (VRPPD) is a VRP in which the possibility that customers return some commodities is contemplated. So in VRPPD it's needed to take into account that the goods that customers return to the delivery vehicle must fit into it. This restriction makes the planning problem more difficult and can lead to bad utilization of the vehicles capacities, increased travel distances or a need for more vehicles.

Hence, it is usually to consider restricted situations where all delivery demands start from the depot and all pick-up demands shall be brought back to the depot, so there are no interchanges of goods between the customers. Another alternative is relaxing the restriction that all customers have to be visited exactly once. Another usual simplification is to consider that every vehicle must deliver all the commodities before picking up any goods.

- **Objective:** The objective is to minimize the vehicle fleet and the sum of travel time, with the restriction that the vehicle must have enough capacity for transporting the commodities to be delivered and those ones picked-up at customers for returning them to the depot.

- **Feasibility:** A solution is feasible if the total quantity assigned to each route does not exceed the capacity of the vehicle which services the route and the vehicle has enough capacity for picking-up the commodities at customers.

- **Formulation:** The cost of a route is like in the case of VRP, with the additional restriction that a route is feasible if and only if it is delivery-feasible, pick-up-feasible, and load-feasible. First of all, we shall define p as a vector of the customer's pick-up demand.
    - Delivery-feasible: this case means that the total amount of commodities to serve in a route must not exceed the vehicle's capacity. Given a route $R_i = \{v_0, v_1, \dots, v_{n+1}\}$ and the vehicle assigned to it with capacity C, this constraint can be mathematically expressed by:

  $C_d(v_k) \le C$ and $C_d(v_{k+1}) \ge C$, where $C_d(v_k)$ is the total quantity of goods delivered to all customers of the path of a route that begins on

v_1 (depot) and finish at v_k: $C_d(v_k) = \sum_{v_i \in P(1,v_k)} d_i$. $P(1,v_k)$ denotes the customers visited along the path from the depot until v_k, including customer v_k.

- Pick-up feasible: This constraint ensures that the vehicle has enough capacity to pick-up the goods of all the customers of the route.

$C_p(v_k) \leq C$ and $C_p(v_{k+1}) > C$ where $C_p(v_k)$ is the total quantity of goods picked-up from all customers along the path of a route up to and including node v_k, that is: $C_p(v_k) = \sum_{v_i \in P(1,v_k)} p_i$.

- Load-feasible: the vehicle's capacity can be violated at any node of the route. Such a violation will depend on the sequence of the customers. Let $L(v_k)$ be the vehicle's load just after leaving customer v_k. Assume that the vehicle leaves the depot with an initial load $L(1) \leq C$.

  Then the vehicle's load at any point of the route is

  $$L(v_k) = C_p(v_k) + L(1) - C_d(i_k)$$

  The vehicle's load given by this equation can exceed the vehicle's capacity. This means that the path becomes infeasible because the vehicle cannot perform service at next customer v_{k+1} on the path. So a route is load feasible if $L(v_k) \leq C$ and $L(v_{k+1} > C)$.

Split Delivery VRP (SDVRP):

SDVRP is a relaxation of the VRP wherein it is allowed that the same customer can be served by different vehicles if it reduces overall costs. This relaxation is very important if the sizes of the customer orders are as big as the capacity of a vehicle. It is more difficult to obtain the optimal solution in the SDVRP that in the VRP.

- **Objective:** The objective is to minimize the vehicle fleet and the sum of travel time needed to supply all customers.

- **Feasibility:** A solution is feasible if all constraints of VRP are satisfied except that a customer may be supplied by more than one vehicle.

- **Formulation:** Minimize the sum of the cost of all routes. An easy way to transform a VRP into a SDVRP consists on allowing split deliveries by splitting each customer order into a number of smaller indivisible orders [Burrows 1988].

Stochastic VRP (SVRP):

Stochastic VRP (SVRP) are VRPs where one or several components of the problem are random. Three different kinds of SVRP are the next examples:

- Stochastic customers: Each customer v_i is present with probability and p_i absent with probability . $1 - p_i$

- Stochastic demands: The demand of d_i each customer is a random variable.
- Stochastic times: Service times and δ_i travel times are t_{ij} random variables.

In SVRP, two stages are made for getting a solution. A first solution is determined before knowing the realizations of the random variables. In a second stage, a recourse or corrective action can be taken when the values of the random variables are known.

- **Objective:** The objective is to minimize the vehicle fleet and the sum of travel time needed to supply all customers with random values on each execution for the customers to be served, their demands and/or the service and travel times.

- **Feasibility:** When some data are random, it is no longer possible to require that all constraints be satisfied for all realizations of the random variables. So the decision maker may either require the satisfaction of some constraints with a given probability, or the incorporation into the model of corrective actions to be taken when a constraint is violated.

- **Formulation:** Minimize $\sum_{i<j} c_{ij} x_{ij} + Q(x)$, where:
    - x_{ij} is an integer variable equal to the number of times edge (v_i, v_j) appears in the first stage solution. If $i, j > 1$, then x_{ij} can only take the values 0 or 1, if $i = 1$, x_{ij} be equal to 2 if a vehicle makes a return trip between the depot and v_j.
    - $Q(x)$ is the expected second stage recourse function. It is problem dependent and is also related to the particular choice of possible recourse actions. For example, in the capacity constrained SVRP with collections, possible recourse actions are:
        - Return to the depot when the vehicle is full in order to unload, and then resume collections as planned.
        - Return to the depot when the vehicle is full as in the previous case and reoptimize the remaining part of the planned route.
        - Planning a preventive return to the depot even if the vehicle is not full. In such case, this decision could depend on the amount already collected and on the distance separating the vehicle from the depot.

A vehicle not yet full may return to the depot if it is known that going to the next customer would cause its capacity to be exceeded.

Periodic VRP (PVRP):

In classical VRPs, typically the planning period is a single day. In the case of the Period Vehicle Routing Problem (PVRP), the classical VRP is generalized by extending the planning period to M days.

We define the problem as follows:

- **Objective:** The objective is to minimize the vehicle fleet and the sum of travel time needed to supply all customers.

- **Feasibility:** A solution is feasible if all constraints of VRP are satisfied. Furthermore a vehicle may not return to the depot in the same day it departs. Over the M-day period, each customer must be visited at least once.

- **Formulation:** Minimize the sum of the cost of all routes. Each customer has a known daily demand that must be completely satisfied in only one visit by exactly one vehicle. If the planning period M = 1, then PVRP becomes an instance of the classical VRP. Each customer in PVRP must be visited k times, where $1 \leq k \leq M$. In the classical model of PVRP, the daily demand of a customer is always fixed. The PVRP can be seen as a problem of generating a group of routes for each day so that the constraints involved are satisfied and the global costs are minimized. PVRP can also be seen as a multi-level combinatorial optimization problem:
    - In the first level, the objective is to generate a group of feasible alternatives (combinations) for each customer. For example, if the planning period has $d = 3$ days then the possible combinations are: $\{d1, d2, d3\}$;; ; ; ; ;and . If a customer requests two visits, then this customer has the following visiting alternatives: $0 \rightarrow 000$ $1 \rightarrow 001$ $2 \rightarrow 010$ $3 \rightarrow 011$ $4 \rightarrow 100$ $5 \rightarrow 101$ $6 \rightarrow 110$ $7 \rightarrow 111$

$\{d1, d2\}$

$\{d1, d3\}$ $\{d2, d3\}$

, and (or the options: 3, 5 and 6 of the table below).

Customer	Diary Demand	Number of Visits	Number of Combinations	Possible Combinations
1	30	1	3	1, 2, 4
2	20	2	3	3, 5, 6
3	20	2	3	3, 5, 6
4	30	2	3	1, 2, 4
5	10	3	1	7

- In the second level, we must select one of the alternatives for each customer, so that the daily constraints are satisfied. Thus we must select the customers to be visited in each day.
- In the third level, we solve the vehicle routing problem for each day.

Solution Techniques for VRP:

Almost all of the techniques that are used for solving Vehicle Routing Problems are heuristics and metaheuristics because no exact algorithm can be guaranteed to find optimal tours within reasonable computing time when the number of cities is large. This is due to the NP-Hardness of the problem.

The following are the most common of these techniques with the algorithms and methods that belong to each of them:

1 - Exact Approaches:

As the name suggests, this approach proposes to compute every possible solution until one of the bests is reached.

- Branch and bound (up to 100 nodes) (Fisher 1994)
- Branch and cut

2 - Heuristics:

Heuristic methods perform a relatively limited exploration of the search space and typically produce good quality solutions within modest computing times.

- Constructive Methods:

  Gradually build a feasible solution while keeping an eye on solution cost, but do not contain an improvement phase *per se*.

  - Savings: Clark and Wright (1964)
  - Matching Based
  - Multi-route Improvement Heuristics
    - Thompson and Psaraftis (1993)
    - Van Breedam (1994)
    - Kinderwater and Savelsbergh (1997)
- 2-Phase Algorithm:

  The problem is decomposed into its two natural components:

1. Clustering of vertices into feasible routes
2. Actual route construction

There are possible feedback loops between the two stages.

- Cluster-First, Route-Second Algorithms
    - Fisher and Jaikumar (1981)
    - The Petal Algorithm
    - The Sweep Algorithm
    - Taillard (1993)
- Route-First, Cluster-Second Algorithms

3 - Meta-Heuristics:

In metaheuristics, the emphasis is on performing a deep exploration of the most promising regions of the solution space. The quality of solutions produced by these methods is much higher than that obtained by classical heuristics.

- Ant Algorithms
- Constraint Programming
- Deterministic Annealing
- Genetic Algorithms
- Simulated Annealing

- Tabu Search
    - Granular Tabu
    - The adaptative memory procedure
    - Kelly and Xu (1999)

The Implementation:

I used Borland Delphi programming language to design a program that tries to simulate Capacitated Vehicle Routing Problem. In this program, we enter the parameters which are: the number of the nodes or customers, the costs of travelling between these nodes, the demands of these nodes or customers, the vehicle’s capacity, and the number of the vehicles. The program runs and tells us if there is a feasible solution or not based on the three major conditions of the CVRP which are: starting and ending at the depot 0, visiting each node or customer exactly once, and not exceeding the capacity of the vehicle. If there is a feasible solution, the program allows us to see the generated routes.

Conclusion:

During my working on this topic, I have read many materials. Most of them were online articles. This reading really improved my knowledge not only about the Vehicle Routing Problem but also about the entire idea of Computational Complexity. Before taking this course I had no idea what this term (Computational Complexity) means. I am glad that I took this class because it let me become more familiar with the theoretical classes which I have not taken any of them in my undergraduate study.

I would like to thank Professor Tom who advised me to take this class and I hope that I will have an opportunity to take other classes with him.